

THE WELL-POSEDNESS ISSUE FOR THE DENSITY-DEPENDENT EULER EQUATIONS IN ENDPOINT BESOV SPACES

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ABSTRACT. This work is the continuation of the recent paper [9] devoted to the density-dependent incompressible Euler equations. Here we concentrate on the well-posedness issue in Besov spaces of type $B_{\infty,r}^s$ embedded in the set of Lipschitz continuous functions, a functional framework which contains the particular case of Hölder spaces and of the endpoint Besov space $B_{\infty,1}^1$. For such data and under the nonvacuum assumption, we establish the local well-posedness and a continuation criterion in the spirit of that of Beale, Kato and Majda in [2].

In the last part of the paper, we give lower bounds for the lifespan of a solution. In dimension two, we point out that the lifespan tends to infinity when the initial density tends to be a constant. This is, to our knowledge, the first result of this kind for the density-dependent incompressible Euler equations.

1. INTRODUCTION AND MAIN RESULTS

This work is the continuation of a recent paper by the first author (see [9]) devoted to the *density-dependent incompressible Euler equations*:

$$(1) \quad \begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho(\partial_t u + u \cdot \nabla u) + \nabla \Pi = \rho f, \\ \operatorname{div} u = 0. \end{cases}$$

Recall that the above equations describe the evolution of the density $\rho = \rho(t, x) \in \mathbb{R}_+$ and of the velocity field $u = u(t, x) \in \mathbb{R}^N$ of a nonhomogeneous inviscid incompressible fluid. The time dependent vector field f stands for a given body force and the gradient of the pressure $\nabla \Pi$ is the Lagrangian multiplier associated to the divergence free constraint over the velocity. We assume that the space variable x belongs to the whole \mathbb{R}^N with $N \geq 2$.

There is an important literature devoted to the standard incompressible Euler equations, that is to the case where the initial density is a positive constant, an assumption which is preserved during the evolution. In contrast, not so many works have been devoted to the study of (1) in the nonconstant density case. In the situation where the equations are considered in a suitably smooth bounded domain of \mathbb{R}^2 or \mathbb{R}^3 , the local well-posedness issue has been investigated by H. Beirão da Veiga and A. Valli in [3, 4, 5] for data with high enough Hölder regularity. In [8], we have proved well-posedness in H^s with $s > 1 + N/2$ and have studied the inviscid limit in this framework. Data in the limit Besov space $B_{2,1}^{\frac{N}{2}+1}$ were also considered.

As for the standard incompressible Euler equations, any functional space embedded in the set $C^{0,1}$ of bounded globally Lipschitz functions is a candidate for the study of the well-posedness issue. This stems from the fact that System (1) is a coupling between transport equations. Hence preserving the initial regularity requires the velocity field to be at least locally Lipschitz with respect to the space variable. As a matter of fact, the classical Euler equations have been shown to be well posed in any Besov space $B_{p,r}^s$ embedded in $C^{0,1}$ (see [1, 7, 13, 18] and the references therein), a property which holds if and only if $(s, p, r) \in \mathbb{R} \times [1, +\infty]^2$ satisfies

$$(C) \quad s > 1 + \frac{N}{p} \quad \text{or} \quad s = 1 + \frac{N}{p} \quad \text{and} \quad r = 1.$$

In [9], we extended the results of the homogeneous case to (1) (see also [10] for a similar study in the periodic framework). Under condition (C) with $1 < p < \infty$ we established the local well-posedness for any data (ρ_0, u_0) in $B_{p,r}^s$ such that ρ_0 is bounded away from zero. However, we have been unable to treat the limit case $p = \infty$ unless the initial density is a small perturbation of a constant density state, a technical artifact due to the method we used to handle the pressure term.

In fact, in contrast to the classical Euler equations, computing the gradient of the pressure involves an elliptic equation with nonconstant coefficients, namely

$$(2) \quad \operatorname{div}(a\nabla\Pi) = \operatorname{div} F \quad \text{with } F := \operatorname{div}(f - u \cdot \nabla u) \quad \text{and } a := 1/\rho.$$

Getting appropriate a priori estimates given that we expect the function ρ to have exactly the same regularity as $\nabla\Pi$ is the main difficulty. In the L^2 framework and, more generally, in the Sobolev framework H^s , this may be achieved by means of a classical energy method. This is also quite straightforward in the $B_{p,r}^s$ framework if a is a small perturbation of some positive constant function \bar{a} , for the above equation may be rewritten

$$\bar{a}\Delta\Pi = \operatorname{div} F + \operatorname{div}((\bar{a} - a)\nabla\Pi).$$

Now, if $a - \bar{a}$ is small enough then one may take advantage of regularity results for the Laplace operator in order to “absorb” the last term.

If $1 < p < \infty$ and a is bounded away from zero then it turns out that combining energy arguments similar to those of the H^s case and a harmonic analysis lemma allows to handle the elliptic equation (2). This is the approach that we used in [9]. However it fails for the limit cases $p = 1$ and $p = \infty$.

In the present work, we propose another method for proving a priori estimates for (2). In addition to being simpler, this will enable us to treat all the cases $p \in [1, \infty]$ indistinctly whenever the density is bounded away from zero. Our approach relies on the fact that the pressure Π satisfies (here we take $f \equiv 0$ to simplify)

$$(3) \quad \Delta\Pi = -\rho \operatorname{div}(u \cdot \nabla u) + \nabla \log \rho \cdot \nabla\Pi.$$

Obviously, the last term is of lower order. In addition, the classical L^2 theory ensures that

$$a_* \|\nabla\Pi\|_{L^2} \leq \|u \cdot \nabla u\|_{L^2} \quad \text{with } a_* := \inf_{x \in \mathbb{R}^N} a(x).$$

Therefore interpolating between the high regularity estimates for the Laplace operator and the L^2 estimate allows to absorb the last term in the right-hand side of (3).

In the rest of the paper, we focus on the case $p = \infty$ as it is the only definitely new one and as it covers both Hölder spaces with exponent greater than 1 and the limit space $B_{\infty,1}^1$ which is the largest one in which one may expect to get well-posedness.

Before going further into the description of our results, let us introduce a few notation.

- Throughout the paper, C stands for a harmless “constant” the meaning of which depends on the context.
- If $a = (a^1, a^2)$ and $b = (b^1, b^2)$ then we denote $a \wedge b := a^1 b^2 - a^2 b^1$.
- The vorticity Ω associated to a vector field u over \mathbb{R}^N is the matrix valued function with entries

$$\Omega_{ij} := \partial_j u^i - \partial_i u^j.$$

If $N = 2$ then the vorticity may be identified with the scalar function $\omega := \partial_1 u^2 - \partial_2 u^1$ and if $N = 3$, with the vector field $\nabla \times u$.

- For all Banach space X and interval I of \mathbb{R} , we denote by $\mathcal{C}(I; X)$ (resp. $\mathcal{C}_b(I; X)$) the set of continuous (resp. continuous bounded) functions on I with values in X . If X has predual X^* then we denote by $\mathcal{C}_w(I; X)$ the set of bounded measurable functions $f : I \rightarrow X$ such that for any $\phi \in X^*$, the function $t \mapsto \langle f(t), \phi \rangle_{X \times X^*}$ is continuous over I .

- For $p \in [1, \infty]$, the notation $L^p(I; X)$ stands for the set of measurable functions on I with values in X such that $t \mapsto \|f(t)\|_X$ belongs to $L^p(I)$. In the case $I = [0, T]$ we alternately use the notation $L^p_T(X)$.
- We denote by $L^p_{loc}(I)$ the set of those functions defined on I and valued in X which, restricted to any compact subset J of I , are in $L^p(J)$.
- Finally, for any real valued function a over \mathbb{R}^N , we denote

$$a_* := \inf_{x \in \mathbb{R}^N} a(x) \quad \text{and} \quad a^* := \sup_{x \in \mathbb{R}^N} a(x).$$

Let us now state our main well-posedness result in the case of a finite energy initial velocity field.

Theorem 1. *Let r be in $[1, \infty]$ and $s \in \mathbb{R}$ satisfy $s > 1$ if $r \neq 1$ and $s \geq 1$ if $r = 1$. Let ρ_0 be a positive function in $B_{\infty, r}^s$ bounded away from 0, and u_0 be a divergence-free vector field with coefficients in $B_{\infty, r}^s \cap L^2$. Finally, suppose that the external force f has coefficients in $L^1([-T_0, T_0]; B_{\infty, r}^s) \cap \mathcal{C}([-T_0, T_0]; L^2)$ for some positive time T_0 .*

Then there exists a time $T \in]0, T_0]$ such that System (1) with initial data (ρ_0, u_0) has a unique solution $(\rho, u, \nabla \Pi)$ on $[-T, T] \times \mathbb{R}^N$, with:

- ρ in $\mathcal{C}([-T, T]; B_{\infty, r}^s)$ and bounded away from 0,
- u in $\mathcal{C}([-T, T]; B_{\infty, r}^s) \cap \mathcal{C}^1([-T, T]; L^2)$ and
- $\nabla \Pi$ in $L^1([-T, T]; B_{\infty, r}^s) \cap \mathcal{C}([-T, T]; L^2)$.

If $r = \infty$ then one has only weak continuity in time with values in the Besov space $B_{\infty, \infty}^s$.

In the above functional framework, one may state a continuation criterion for the solution to (1) similar to that of Theorem 2 of [9]:

Theorem 2. *Let $(\rho, u, \nabla \Pi)$ be a solution to System (1) on $[0, T^*] \times \mathbb{R}^N$, with the properties described in Theorem 1 for all $T < T^*$; suppose also that we have*

$$(4) \quad \int_0^{T^*} \left(\|\nabla u\|_{L^\infty} + \|\nabla \Pi\|_{B_{\infty, r}^{s-1}} \right) dt < \infty.$$

If T^ is finite then $(\rho, u, \nabla \Pi)$ can be continued beyond T^* into a solution of (1) with the same regularity. Moreover, if $s > 1$ then one may replace in (4) the term $\|\nabla u\|_{L^\infty}$ with $\|\Omega\|_{L^\infty}$.*

A similar result holds for negative times.

From this result, as our assumption on (r, s) implies that $B_{\infty, r}^{s-1} \hookrightarrow L^\infty$, keeping in mind that $B_{\infty, 1}^1$ is the largest Besov space included in $C^{0,1}$, we immediately get the following:

Corollary 1. *The lifespan of a solution in $B_{\infty, r}^s$ with $s > 1$ is the same as the lifespan in $B_{\infty, 1}^1$.*

As pointed out in [9], hypothesis $u_0 \in L^2$ is somewhat restrictive in dimension $N = 2$ as if, say, the initial vorticity ω_0 is in L^1 then it implies that ω_0 has average 0 over \mathbb{R}^2 . In particular, assuming that $u_0 \in L^2(\mathbb{R}^2)$ precludes our considering general data with initially compactly supported nonnegative vorticity (e.g. vortex patches as in [7], Chapter 5).

The following statement aims at considering initial data *with infinite energy*. For simplicity, we suppose the external force to be 0.

Theorem 3. *Let (s, r) be as in Theorem 1. Let $\rho_0 \in B_{\infty, r}^s$ be bounded away from 0, and $u_0 \in B_{\infty, r}^s \cap W^{1,4}$.*

Then there exist a positive time T and a unique solution $(\rho, u, \nabla \Pi)$ on $[-T, T] \times \mathbb{R}^N$ of System (1) with external force $f \equiv 0$, satisfying the following properties:

- $\rho \in \mathcal{C}([-T, T]; B_{\infty, r}^s)$ bounded away from 0,
- $u \in \mathcal{C}([-T, T]; B_{\infty, r}^s \cap W^{1,4})$ and $\partial_t u \in \mathcal{C}([-T, T]; L^2)$,
- $\nabla \Pi \in L^1([-T, T]; B_{\infty, r}^s) \cap \mathcal{C}([-T, T]; L^2)$.

As above, the continuity in time with values in $B_{\infty, r}^s$ is only weak if $r = \infty$.

Remark 1. *Under the above hypothesis, a continuation criterion in the spirit of Theorem 2 may be proved. The details are left to the reader.*

Let us also point out that there is some freedom over the $W^{1,4}$ assumption (see Remark 4 below).

On the one hand, the existence results that we stated so far are *local in time* even in the two-dimensional case. On the other hand, it is well known that the classical two-dimensional incompressible Euler equations are globally well-posed, a result that goes back to the pioneering work by V. Wolibner in [16] (see also [17, 11, 15] for global results in the case of less regular data). In the homogeneous case, the global existence stems from the fact that the vorticity ω is transported by the flow associated to the solution: we have

$$\partial_t \omega + u \cdot \nabla \omega = 0.$$

In the nonhomogeneous context this relation is not true any longer: we have instead

$$(5) \quad \partial_t \omega + u \cdot \nabla \omega + \nabla \left(\frac{1}{\rho} \right) \wedge \nabla \Pi = 0.$$

If the classical homogeneous case has been deeply studied, to our knowledge there is no literature about the time of existence of solutions for the density-dependent incompressible Euler system. In the last section of this paper, we establish lower bounds for the lifespan of a solution of (1).

Roughly, we show that in any space dimension, if the initial velocity is of order ε (ε small enough), *without any restriction on the density of the fluid* then the lifespan is at least of order ε^{-1} (see the exact statement in Theorem 4).

Next, taking advantage of Equality (5) and of an estimate for the transport equation that has been established recently by M. Vishik in [15] (and generalized by T. Hmidi and S. Keraani in [12]), we show that the lifespan of the solution tends to infinity if $\rho_0 - 1$ goes to 0. More precisely, Theorem 5 states that if

$$\|\rho_0 - 1\|_{B_{\infty,1}^1} = \eta \quad \text{and} \quad \|\omega_0\|_{B_{\infty,1}^0} + \|u_0\|_{L^2} = \varepsilon$$

with η small enough, then the lifespan is at least of order $\varepsilon^{-1} \log(\log \eta^{-1})$.

The paper is organized as follows. In the next section, we introduce the tools needed for proving our results: the Littlewood-Paley decomposition, the definition of the nonhomogeneous Besov spaces $B_{p,r}^s$ and the paradifferential calculus, and finally some classical results about transport equations in $B_{p,r}^s$ and elliptic equations. Sections 3 and 4 are devoted to the proof of our local existence statements first in the finite energy case and next if the initial velocity is in $W^{1,4}$. Finally, in the last section we state and prove results about the lifespan of a solution of our system, focusing on the particular case of space dimension $N = 2$.

2. TOOLS

Our results mostly rely on Fourier analysis methods based on a nonhomogeneous dyadic partition of unity with respect to the Fourier variable, the so-called Littlewood-Paley decomposition. Unless otherwise specified, all the results which are presented in this section are proved in [1].

In order to define a Littlewood-Paley decomposition, fix a smooth radial function χ supported in (say) the ball $B(0, \frac{4}{3})$, equals to 1 in a neighborhood of $B(0, \frac{3}{4})$ and such that $r \mapsto \chi(r e_r)$ is nonincreasing over \mathbb{R}_+ , and set $\varphi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi)$.

The *dyadic blocks* $(\Delta_j)_{j \in \mathbb{Z}}$ are defined by¹

$$\Delta_j := 0 \quad \text{if } j \leq -2, \quad \Delta_{-1} := \chi(D) \quad \text{and} \quad \Delta_j := \varphi(2^{-j} D) \quad \text{if } j \geq 0.$$

¹Throughout we agree that $f(D)$ stands for the pseudo-differential operator $u \mapsto \mathcal{F}^{-1}(f\mathcal{F}u)$.

We also introduce the following low frequency cut-off:

$$S_j u := \chi(2^{-j} D) = \sum_{j' \leq j-1} \Delta_{j'} \quad \text{for } j \geq 0.$$

The following classical properties will be used freely throughout in the paper:

- for any $u \in \mathcal{S}'$, the equality $u = \sum_j \Delta_j u$ holds true in \mathcal{S}' ;
- for all u and v in \mathcal{S}' , the sequence $(S_{j-1} u \Delta_j v)_{j \in \mathbb{N}}$ is spectrally supported in dyadic annuli.

One can now define what a Besov space $B_{p,r}^s$ is:

Definition 1. Let u be a tempered distribution, s a real number, and $1 \leq p, r \leq \infty$. We set

$$\|u\|_{B_{p,r}^s} := \left(\sum_j 2^{rjs} \|\Delta_j u\|_{L^p}^r \right)^{\frac{1}{r}} \quad \text{if } r < \infty \quad \text{and} \quad \|u\|_{B_{p,\infty}^s} := \sup_j \left(2^{js} \|\Delta_j u\|_{L^p} \right).$$

We then define the space $B_{p,r}^s$ as the subset of distributions $u \in \mathcal{S}'$ such that $\|u\|_{B_{p,r}^s}$ is finite.

From the above definition, it is easy to show that for all $s \in \mathbb{R}$, the Besov space $B_{2,2}^s$ coincides with the nonhomogeneous Sobolev space H^s . Let us also point out that for any $k \in \mathbb{N}$ and $p \in [1, \infty]$, we have the following chain of continuous embedding:

$$B_{p,1}^k \hookrightarrow W^{k,p} \hookrightarrow B_{p,\infty}^k.$$

where $W^{k,p}$ denotes the set of L^p functions with derivatives up to order k in L^p .

The Besov spaces have many nice properties which will be recalled throughout the paper whenever they are needed. For the time being, let us just recall that if Condition (C) holds true then $B_{p,r}^s$ is an algebra continuously embedded in the set $C^{0,1}$ of bounded Lipschitz functions (see e.g. [1], Chap. 2), and that the gradient operator maps $B_{p,r}^s$ in $B_{p,r}^{s-1}$.

The following result will be also needed:

Proposition 1. Let $F : \mathbb{R}^N \rightarrow \mathbb{R}$ be a smooth homogeneous function of degree m away from a neighborhood of the origin. Then for all $(p, r) \in [1, \infty]^2$ and $s \in \mathbb{R}$, Operator $F(D)$ maps $B_{p,r}^s$ in $B_{p,r}^{s-m}$.

Remark 2. Let \mathcal{P} be the Leray projector over divergence free vector fields and $\mathcal{Q} := \text{Id} - \mathcal{P}$. Recall that in Fourier variables, we have for all vector field u

$$\widehat{\mathcal{Q}u}(\xi) = -\frac{\xi}{|\xi|^2} \xi \cdot \widehat{u}(\xi).$$

Therefore, both $(\text{Id} - \Delta_{-1})\mathcal{P}$ and $(\text{Id} - \Delta_{-1})\mathcal{Q}$ satisfy the assumptions of the above proposition with $m = 0$ hence are self-map on $B_{p,r}^s$ for any $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$.

The following lemma (referred in what follows as *Bernstein's inequalities*) describes the way derivatives act on spectrally localized functions.

Lemma 1. Let $0 < r < R$. A constant C exists so that, for any nonnegative integer k , any couple (p, q) in $[1, \infty]^2$ with $q \geq p \geq 1$ and any function u of L^p , we have for all $\lambda > 0$,

$$\begin{aligned} \text{Supp } \widehat{u} \subset B(0, \lambda R) &\implies \|\nabla^k u\|_{L^q} \leq C^{k+1} \lambda^{k+N(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}; \\ \text{Supp } \widehat{u} \subset \{\xi \in \mathbb{R}^N / r\lambda \leq |\xi| \leq R\lambda\} &\implies C^{-k-1} \lambda^k \|u\|_{L^p} \leq \|\nabla^k u\|_{L^p} \leq C^{k+1} \lambda^k \|u\|_{L^p}. \end{aligned}$$

The first Bernstein inequality entails the following embedding result:

Corollary 2. The space $B_{p_1, r_1}^{s_1}$ is continuously embedded in the space $B_{p_2, r_2}^{s_2}$ whenever $1 \leq p_1 \leq p_2 \leq \infty$ and

$$s_2 < s_1 - N/p_1 + N/p_2 \quad \text{or} \quad s_2 = s_1 - N/p_1 + N/p_2 \quad \text{and} \quad 1 \leq r_1 \leq r_2 \leq \infty.$$

Let us now introduce the paraproduct operator and recall a few nonlinear estimates in Besov spaces. Constructing the paraproduct operator relies on the observation that, formally, any product of two tempered distributions u and v , may be decomposed into

$$(6) \quad uv = T_u v + T_v u + R(u, v)$$

with

$$T_u v := \sum_j S_{j-1} u \Delta_j v, \quad T_v u := \sum_j S_{j-1} v \Delta_j u \quad \text{and} \quad R(u, v) := \sum_j \sum_{|j'-j| \leq 1} \Delta_j u \Delta_{j'} v.$$

The above operator T is called ‘‘paraproduct’’ whereas R is called ‘‘remainder’’.

The paraproduct and remainder operators have many nice continuity properties. The following ones will be of constant use in this paper (see the proof in e.g. [1], Chap. 2):

Proposition 2. *For any $(s, p, r) \in \mathbb{R} \times [1, \infty]^2$ and $t < 0$, the paraproduct operator T maps $L^\infty \times B_{p,r}^s$ in $B_{p,r}^s$, and $B_{\infty,\infty}^t \times B_{p,r}^s$ in $B_{p,r}^{s+t}$. Moreover, the following estimates hold:*

$$\|T_u v\|_{B_{p,r}^s} \leq C \|u\|_{L^\infty} \|\nabla v\|_{B_{p,r}^{s-1}} \quad \text{and} \quad \|T_u v\|_{B_{p,r}^{s+t}} \leq C \|u\|_{B_{\infty,\infty}^t} \|\nabla v\|_{B_{p,r}^{s-1}}.$$

For any (s_1, p_1, r_1) and (s_2, p_2, r_2) in $\mathbb{R} \times [1, \infty]^2$ such that $s_1 + s_2 > 0$, $1/p := 1/p_1 + 1/p_2 \leq 1$ and $1/r := 1/r_1 + 1/r_2 \leq 1$ the remainder operator R maps $B_{p_1,r_1}^{s_1} \times B_{p_2,r_2}^{s_2}$ in $B_{p,r}^{s_1+s_2}$.

Combining the above proposition with Bony’s decomposition (6), we easily get the following ‘‘tame estimate’’:

Corollary 3. *Let a be a bounded function such that $\nabla a \in B_{p,r}^{s-1}$ for some $s > 0$ and $(p, r) \in [1, \infty]^2$. Then for any $b \in B_{p,r}^s \cap L^\infty$ we have $ab \in B_{p,r}^s \cap L^\infty$ and there exists a constant C depending only on N , p and s such that*

$$\|ab\|_{B_{p,r}^s} \leq C \left(\|a\|_{L^\infty} \|b\|_{B_{p,r}^s} + \|b\|_{L^\infty} \|\nabla a\|_{B_{p,r}^{s-1}} \right).$$

The following result pertaining to the composition of functions in Besov spaces will be needed for estimating the reciprocal of the density (see the proof in [9]).

Proposition 3. *Let I be an open interval of \mathbb{R} and $F : I \rightarrow \mathbb{R}$, a smooth function. Then for all compact subset $J \subset I$, $s > 0$ and $(p, r) \in [1, \infty]^2$ there exists a constant C such that for all function a valued in J and with gradient in $B_{p,r}^{s-1}$, we have $\nabla(F(a)) \in B_{p,r}^{s-1}$ and*

$$\|\nabla(F(a))\|_{B_{p,r}^{s-1}} \leq C \|\nabla a\|_{B_{p,r}^{s-1}}.$$

Our results concerning Equations (1) rely strongly on a priori estimates in Besov spaces for the transport equation

$$(T) \quad \begin{cases} \partial_t a + v \cdot \nabla a = f, \\ a|_{t=0} = a_0. \end{cases}$$

We shall often use the following result, the proof of which may be found in e.g. [1].

Proposition 4. *Let $1 \leq r \leq \infty$ and $\sigma > 0$ ($\sigma > -1$ if $\operatorname{div} v = 0$). Let $a_0 \in B_{\infty,r}^\sigma$, $f \in L^1([0, T]; B_{\infty,r}^\sigma)$ and v be a time dependent vector field in $C_b([0, T] \times \mathbb{R}^N)$ such that*

$$\begin{aligned} \nabla v &\in L^1([0, T]; L^\infty) && \text{if } \sigma < 1, \\ \nabla v &\in L^1([0, T]; B_{\infty,r}^{\sigma-1}) && \text{if } \sigma > 1, \quad \text{or } \sigma = r = 1. \end{aligned}$$

Then Equation (T) has a unique solution a in

- the space $\mathcal{C}([0, T]; B_{\infty,r}^\sigma)$ if $r < \infty$,
- the space $\left(\bigcap_{\sigma' < \sigma} \mathcal{C}([0, T]; B_{\infty,\infty}^{\sigma'}) \right) \cap \mathcal{C}_w([0, T]; B_{\infty,\infty}^\sigma)$ if $r = \infty$.

Moreover, for all $t \in [0, T]$, we have

$$(7) \quad e^{-CV(t)} \|a(t)\|_{B_{\infty,r}^{\sigma}} \leq \|a_0\|_{B_{\infty,r}^{\sigma}} + \int_0^t e^{-CV(t')} \|f(t')\|_{B_{\infty,r}^{\sigma}} dt'$$

$$\text{with } V'(t) := \begin{cases} \|\nabla v(t)\|_{L^{\infty}} & \text{if } \sigma < 1, \\ \|\nabla v(t)\|_{B_{\infty,r}^{\sigma-1}} & \text{if } \sigma > 1, \text{ or } \sigma = r = 1. \end{cases}$$

If $a = v$ then, for all $\sigma > 0$ ($\sigma > -1$ if $\operatorname{div} v = 0$), Estimate (7) holds with $V'(t) := \|\nabla a(t)\|_{L^{\infty}}$.

Finally, we shall make an extensive use of energy estimates for the following elliptic equation:

$$(8) \quad -\operatorname{div}(a\nabla\Pi) = \operatorname{div} F \quad \text{in } \mathbb{R}^N$$

where $a = a(x)$ is a given suitably smooth bounded function satisfying

$$(9) \quad a_* := \inf_{x \in \mathbb{R}^N} a(x) > 0.$$

We shall use the following result based on Lax-Milgram's theorem (see the proof in e.g. [9]).

Lemma 2. *For all vector field F with coefficients in L^2 , there exists a tempered distribution Π , unique up to constant functions, such that $\nabla\Pi \in L^2$ and Equation (8) is satisfied. In addition, we have*

$$(10) \quad a_* \|\nabla\Pi\|_{L^2} \leq \|F\|_{L^2}.$$

3. PROOF OF THEOREM 1

Obviously, one may extend the force term for any time so that it is not restrictive to assume that $T_0 = +\infty$. Owing to time reversibility of System (1), we can restrict ourselves to the problem of evolution for positive times only. For convenience we will assume $r < \infty$; for treating the case $r = \infty$, it is enough to replace the strong topology by the weak topology, whenever regularity up to index s is involved.

We will not work on System (1) directly, but rather on

$$(11) \quad \begin{cases} \partial_t a + u \cdot \nabla a = 0 \\ \partial_t u + u \cdot \nabla u + a\nabla\Pi = f \\ -\operatorname{div}(a\nabla\Pi) = \operatorname{div}(u \cdot \nabla \mathcal{P}u - f), \end{cases}$$

where we have set $a := 1/\rho$.

The equivalence between (1) and (11) is given in the following statement (see [9]).

Lemma 3. *Let u be a vector field with coefficients in $\mathcal{C}^1([0, T] \times \mathbb{R}^N)$ and such that $\mathcal{Q}u \in \mathcal{C}^1([0, T]; L^2)$. Suppose also that $\nabla\Pi \in \mathcal{C}([0, T]; L^2)$. Finally, let ρ be a continuous function on $[0, T] \times \mathbb{R}^N$ such that*

$$(12) \quad 0 < \rho_* \leq \rho \leq \rho^*.$$

Let $a := 1/\rho$. If $\operatorname{div} u(0, \cdot) \equiv 0$ in \mathbb{R}^N then $(\rho, u, \nabla\Pi)$ is a solution to (1) if and only if $(a, u, \nabla\Pi)$ is a solution to (11).

We now come to the plan of this section. First of all, we shall prove a priori estimates for suitably smooth solutions of (1) or (11). Even though those estimates are not needed for proving Theorem 1, they will be most helpful to get the existence. As a matter of fact, the construction of solutions which will be proposed in the next subsection amounts to solving inductively a sequence of *linear* equations. The estimates for those approximate solutions turn out to be the same as those for the true solutions. In the last two subsections, we shall concentrate on the proof of the uniqueness part of Theorem 1 and of the continuation criterion stated in Theorem 2 (up to the endpoint case $s = r = 1$ which will be studied in the next section).

3.1. A priori estimates. Let $(a, u, \nabla \Pi)$ be a suitably smooth solution of System (11) with the required regularity properties. In this subsection, we show that on a suitably small time interval (the length of which depends only on the norms of the data), the norm of $(a, u, \nabla \Pi)$ may be bounded in terms of the data.

Recall that according to Proposition 3 the quantities $\|a\|_{B_{\infty,r}^s}$ and $\|\rho\|_{B_{\infty,r}^s}$ are equivalent under hypothesis (12). This fact will be used repeatedly in what follows.

3.1.1. Estimates for the density and the velocity field. Let us assume for a while that $\operatorname{div} u = 0$. Then $(\rho, u, \nabla \Pi)$ satisfies System (1) and the following energy equality holds true:

$$(13) \quad \|\sqrt{\rho(t)} u(t)\|_{L^2}^2 = \|\sqrt{\rho_0} u_0\|_{L^2}^2 + 2 \int_0^t \left(\int_{\mathbb{R}^N} \rho f \cdot u \, dx \right) d\tau.$$

Moreover, from the equation satisfied by the density, we have that $\rho(t, x) = \rho_0(\psi_t^{-1}(x))$, where ψ is the flow associated with u ; so, ρ satisfies (12). Hence, from relation (13), we obtain the control of the L^2 norm of the velocity field: for all $t \in [0, T_0]$, we have, for some constant C depending only on ρ_* and ρ^* ,

$$(14) \quad \|u(t)\|_{L^2} \leq C \left(\|u_0\|_{L^2} + \int_0^t \|f(\tau)\|_{L^2} d\tau \right).$$

Next, in the general case where $\operatorname{div} u$ need not be 0, applying Proposition 4 yields the following estimates:

$$(15) \quad \|a(t)\|_{B_{\infty,r}^s} \leq \|a_0\|_{B_{\infty,r}^s} \exp \left(C \int_0^t \|u\|_{B_{\infty,r}^s} d\tau \right)$$

$$(16) \quad \|u(t)\|_{B_{\infty,r}^s} \leq \exp \left(C \int_0^t \|u\|_{B_{\infty,r}^s} d\tau \right) \cdot \left(\|u_0\|_{B_{\infty,r}^s} + \int_0^t e^{-C \int_0^\tau \|u\|_{B_{\infty,r}^s} d\tau'} \left(\|f\|_{B_{\infty,r}^s} + \|a\|_{B_{\infty,r}^s} \|\nabla \Pi\|_{B_{\infty,r}^s} \right) d\tau' \right),$$

where, in the last line, we have used the fact that $B_{\infty,r}^s$, under our hypothesis, is an algebra.

Remark 3. *Of course, as ρ and a verify the same equations, they satisfy the same estimates.*

3.1.2. Estimates for the pressure term. Let us use the low frequency localization operator Δ_{-1} to separate $\nabla \Pi$ into low and high frequencies. We get

$$\|\nabla \Pi\|_{B_{\infty,r}^s} \leq \|\Delta_{-1} \nabla \Pi\|_{B_{\infty,r}^s} + \|(\operatorname{Id} - \Delta_{-1}) \nabla \Pi\|_{B_{\infty,r}^s}.$$

Observe that $(\operatorname{Id} - \Delta_{-1}) \nabla \Pi$ may be computed from $\Delta \Pi$ by means of a homogeneous multiplier of degree -1 in the sense of Proposition 1. Hence

$$(17) \quad \|(\operatorname{Id} - \Delta_{-1}) \nabla \Pi\|_{B_{\infty,r}^s} \leq C \|\Delta \Pi\|_{B_{\infty,r}^{s-1}}.$$

For the low frequencies term, however, the above inequality fails. Now, remembering the definition of $\|\cdot\|_{B_{\infty,r}^s}$ and the spectral properties of operator Δ_{-1} , one has that

$$\|\Delta_{-1} \nabla \Pi\|_{B_{\infty,r}^s} \leq C \|\Delta_{-1} \nabla \Pi\|_{L^\infty};$$

at this point, Bernstein's inequality allows us to write that

$$\|\Delta_{-1} \nabla \Pi\|_{B_{\infty,r}^s} \leq C \|\nabla \Pi\|_{L^2}.$$

So putting together (17) and the above inequality, we obtain

$$(18) \quad \|\nabla \Pi\|_{B_{\infty,r}^s} \leq C \left(\|\nabla \Pi\|_{L^2} + \|\Delta \Pi\|_{B_{\infty,r}^{s-1}} \right).$$

First of all, let us see how to control $\|\Delta \Pi\|_{B_{\infty,r}^{s-1}}$. Recall the third equation of (11):

$$\operatorname{div}(a \nabla \Pi) = F \quad \text{with} \quad F := \operatorname{div}(f - u \cdot \nabla \mathcal{P}u).$$

Developing the left-hand side of this equation, we obtain

$$(19) \quad \Delta \Pi = -\nabla(\log a) \cdot \nabla \Pi + \frac{F}{a}.$$

Let us consider the first term of the right-hand side of the previous equation.

If $s > 1$ then one may use that $B_{\infty,r}^{s-1}$ is an algebra and bound $\|\nabla(\log a)\|_{B_{\infty,r}^{s-1}}$ with $\|\nabla a\|_{B_{\infty,r}^{s-1}}$ according to Proposition 3; we get

$$\|\nabla(\log a) \cdot \nabla \Pi\|_{B_{\infty,r}^{s-1}} \leq C \|\nabla a\|_{B_{\infty,r}^{s-1}} \|\nabla \Pi\|_{B_{\infty,r}^{s-1}}.$$

Now, as $L^2 \hookrightarrow B_{\infty,\infty}^{-\frac{N}{2}}$ (see Corollary 2) and $B_{\infty,r}^{s-1}$ is an intermediate space between $B_{\infty,\infty}^{-\frac{N}{2}}$ and $B_{\infty,r}^s$, standard interpolation inequalities (see e.g. [1], Chap. 2) ensure that

$$(20) \quad \|\nabla \Pi\|_{B_{\infty,r}^{s-1}} \leq C \|\nabla \Pi\|_{L^2}^\theta \|\nabla \Pi\|_{B_{\infty,r}^s}^{1-\theta} \quad \text{for some } \theta \in]0, 1[.$$

Plugging this inequality in (18) and applying Young's inequality, we finally obtain

$$(21) \quad \|\nabla \Pi\|_{B_{\infty,r}^s} \leq C \left(\left(1 + \|\nabla a\|_{B_{\infty,r}^{s-1}}^\gamma \right) \|\nabla \Pi\|_{L^2} + \left\| \frac{F}{a} \right\|_{B_{\infty,r}^{s-1}} \right),$$

where the exponent γ depends only on the space dimension N and on s .

In the limit case $s = r = 1$, the space $B_{\infty,1}^{s-1}$ is no more an algebra and we have to modify the above argument: we use the Bony decomposition (6) to write

$$\nabla(\log a) \cdot \nabla \Pi = T_{\nabla(\log a)} \nabla \Pi + T_{\nabla \Pi} \nabla(\log a) + R(\nabla(\log a), \nabla \Pi).$$

To estimate first and second term, we can apply Propositions 2 and 3: we get

$$(22) \quad \begin{aligned} \|T_{\nabla(\log a)} \nabla \Pi\|_{B_{\infty,1}^0} &\leq C \|\nabla(\log a)\|_{L^\infty} \|\nabla \Pi\|_{B_{\infty,1}^0} \\ &\leq C \|\nabla a\|_{L^\infty} \|\nabla \Pi\|_{B_{\infty,1}^0}, \end{aligned}$$

$$(23) \quad \begin{aligned} \|T_{\nabla \Pi} \nabla(\log a)\|_{B_{\infty,1}^0} &\leq C \|\nabla \Pi\|_{L^\infty} \|\nabla(\log a)\|_{B_{\infty,1}^0} \\ &\leq C \|\nabla \Pi\|_{L^\infty} \|\nabla a\|_{B_{\infty,1}^0}. \end{aligned}$$

A similar inequality is no more true for the remainder term, though. However, one may use that $\nabla \Pi$ is in fact more regular: it belongs to $B_{\infty,1}^{\frac{1}{2}}$ for instance. Hence, using the embedding $B_{\infty,1}^{\frac{1}{2}} \hookrightarrow B_{\infty,1}^0$ and Proposition 2, we can write

$$\begin{aligned} \|R(\nabla(\log a), \nabla \Pi)\|_{B_{\infty,1}^0} &\leq C \|\nabla(\log a)\|_{L^\infty} \|\nabla \Pi\|_{B_{\infty,1}^{\frac{1}{2}}}, \\ &\leq C \|\nabla a\|_{L^\infty} \|\nabla \Pi\|_{B_{\infty,1}^{\frac{1}{2}}}. \end{aligned}$$

Putting the above inequality together with (22) and (23), and using that $B_{\infty,1}^0 \hookrightarrow L^\infty$, we conclude that

$$\|\nabla(\log a) \cdot \nabla \Pi\|_{B_{\infty,1}^0} \leq C \|\nabla a\|_{B_{\infty,1}^0} \|\nabla \Pi\|_{B_{\infty,1}^{\frac{1}{2}}}.$$

Now, using interpolation between Besov spaces, as done for proving (20), we get for some suitable $\theta \in]0, 1[$,

$$\|\nabla(\log a) \cdot \nabla \Pi\|_{B_{\infty,1}^0} \leq C \|\nabla a\|_{B_{\infty,1}^0} \|\nabla \Pi\|_{B_{\infty,1}^1}^{1-\theta} \|\nabla \Pi\|_{L^2}^\theta.$$

Hence $\|\nabla \Pi\|_{B_{\infty,1}^1}$ satisfies Inequality (21) for some convenient $\gamma > 0$.

Next, let us bound the last term of (19). By virtue of Bony's decomposition (6), we have

$$\frac{F}{a} = \rho F = T_\rho F + T_F \rho + R(\rho, F);$$

so from Proposition 2 we infer that

- $\|T_\rho F\|_{B_{\infty,r}^{s-1}} \leq C \rho^* \|F\|_{B_{\infty,r}^{s-1}},$
- $\|TF\rho\|_{B_{\infty,r}^{s-1}} \leq C \|F\|_{B_{\infty,\infty}^{-1}} \|\rho\|_{B_{\infty,r}^s} \leq C \|F\|_{B_{\infty,r}^{s-1}} \|\rho\|_{B_{\infty,r}^s},$
- $\|R(\rho, F)\|_{B_{\infty,r}^{s-1}} \leq \|R(\rho, F)\|_{B_{\infty,r}^s} \leq C \|\rho\|_{B_{\infty,\infty}^1} \|F\|_{B_{\infty,r}^{s-1}} \leq C \|\rho\|_{B_{\infty,r}^s} \|F\|_{B_{\infty,r}^{s-1}}.$

It is clear that $\|\operatorname{div} f\|_{B_{\infty,r}^{s-1}}$ can be controlled by $\|f\|_{B_{\infty,r}^s}$. For the second term of F we have to take advantage, once again, of Bony's decomposition (6) as follows:

$$\operatorname{div} (u \cdot \nabla \mathcal{P}u) = \sum_{i,j} \partial_i u^j \partial_j (\mathcal{P}u)^i = \sum_{i,j} \left(T_{\partial_i u^j} \partial_j \mathcal{P}u^i + T_{\partial_j \mathcal{P}u^i} \partial_i u^j + \partial_i R(u^j, \partial_j \mathcal{P}u^i) \right),$$

where in the last equality we have used also the fact that $\operatorname{div} \mathcal{P}u = 0$. Now, for all i and j we have:

$$\begin{aligned} \left\| T_{\partial_i u^j} \partial_j \mathcal{P}u^i \right\|_{B_{\infty,r}^{s-1}} &\leq C \|\nabla u\|_{L^\infty} \|\nabla \mathcal{P}u\|_{B_{\infty,r}^{s-1}} \\ \left\| T_{\partial_j \mathcal{P}u^i} \partial_i u^j \right\|_{B_{\infty,r}^{s-1}} &\leq C \|\nabla \mathcal{P}u\|_{L^\infty} \|\nabla u\|_{B_{\infty,r}^{s-1}} \\ \left\| \partial_i R(u^j, \partial_j \mathcal{P}u^i) \right\|_{B_{\infty,r}^{s-1}} &\leq \left\| R(u^j, \partial_j \mathcal{P}u^i) \right\|_{B_{\infty,r}^s} \\ &\leq C \|u\|_{B_{\infty,r}^s} \|\nabla \mathcal{P}u\|_{B_{\infty,\infty}^0} \\ &\leq C \|u\|_{B_{\infty,r}^s} \|\nabla \mathcal{P}u\|_{B_{\infty,r}^{s-1}}. \end{aligned}$$

Because, by embedding,

$$(24) \quad \|\nabla \mathcal{P}u\|_{L^\infty} \leq C \|\nabla \mathcal{P}u\|_{B_{\infty,r}^{s-1}},$$

we thus have

$$\|\operatorname{div} (u \cdot \nabla \mathcal{P}u)\|_{B_{\infty,r}^{s-1}} \leq C \|u\|_{B_{\infty,r}^s} \|\nabla \mathcal{P}u\|_{B_{\infty,r}^{s-1}}.$$

In order to bound $\mathcal{P}u$, let us decompose it into low and high frequencies as follows:

$$\mathcal{P}u = \Delta_{-1} \mathcal{P}u + (\operatorname{Id} - \Delta_{-1}) \mathcal{P}u.$$

On the one hand, combining Bernstein's inequality and the fact that \mathcal{P} is an orthogonal projector over L^2 yields

$$\|\Delta_{-1} \nabla \mathcal{P}u\|_{L^\infty} \leq C \|u\|_{L^2}.$$

On the other hand, according to Remark 2, one may write that

$$\|(\operatorname{Id} - \Delta_{-1}) \mathcal{P}u\|_{B_{\infty,r}^s} \leq C \|u\|_{B_{\infty,r}^s}.$$

Therefore we get

$$(25) \quad \|\nabla \mathcal{P}u\|_{B_{\infty,r}^{s-1}} \leq C \|u\|_{B_{\infty,r}^s \cap L^2},$$

from which it follows that

$$(26) \quad \left\| \frac{F}{a} \right\|_{B_{\infty,r}^{s-1}} \leq C \|a\|_{B_{\infty,r}^s} \left(\|f\|_{B_{\infty,r}^s} + \|u\|_{B_{\infty,r}^s \cap L^2}^2 \right).$$

It remains us to control $\|\nabla \Pi\|_{L^2}$. Keeping in mind Lemma 2, from the third equation of System (11) and Inequalities (24)–(25), we immediately get

$$\begin{aligned} a_* \|\nabla \Pi\|_{L^2} &\leq \|f\|_{L^2} + \|u \cdot \nabla \mathcal{P}u\|_{L^2} \\ &\leq \|f\|_{L^2} + \|u\|_{L^2} \|\nabla \mathcal{P}u\|_{L^\infty} \\ &\leq \|f\|_{L^2} + C \|u\|_{B_{\infty,r}^s \cap L^2}^2. \end{aligned}$$

Putting all these inequalities together, we finally obtain

$$(27) \quad \|\nabla \Pi\|_{L_t^1(L^2)} \leq C \left(\|f\|_{L_t^1(L^2)} + \int_0^t \|u\|_{B_{\infty,r}^s \cap L^2}^2 d\tau \right)$$

$$(28) \quad \|\nabla \Pi\|_{L_t^1(B_{\infty,r}^s)} \leq C \left(\left(1 + \|\nabla a\|_{L_t^\infty(B_{\infty,r}^{s-1})}^\gamma \right) \|\nabla \Pi\|_{L_t^1(L^2)} + \|a\|_{L_t^\infty(B_{\infty,r}^s)} \left(\|f\|_{L_t^1(B_{\infty,r}^s)} + \int_0^t \|u\|_{B_{\infty,r}^s \cap L^2}^2 d\tau \right) \right).$$

3.1.3. *Final estimate.* First of all, let us fix $T > 0$ so small as to satisfy

$$(29) \quad \exp\left(C \int_0^T \|u\|_{B_{\infty,r}^s} dt\right) \leq 2,$$

a fact that is always possible because of the continuity of u with respect the time variable.

Then, setting

$$\begin{aligned} U(t) &:= \|u(t)\|_{L^2 \cap B_{\infty,r}^s} = \|u(t)\|_{L^2} + \|u(t)\|_{B_{\infty,r}^s} \\ U_0(t) &:= \|u_0\|_{L^2 \cap B_{\infty,r}^s} + \int_0^t \|f\|_{L^2 \cap B_{\infty,r}^s} d\tau \end{aligned}$$

and combining estimates (14), (15), (16), (27) and (28), we get

$$(30) \quad U(t) \leq C \left(U_0(t) + \int_0^t U^2(\tau) d\tau \right) \quad \text{for all } t \in [0, T],$$

where the constant C depends only on s , N , $\|a_0\|_{B_{\infty,r}^s}$, a_* and a^* .

So, taking T small enough and changing once more the multiplying constant if needed, a standard bootstrap argument allows to show that

$$U(t) \leq C U_0(t) \quad \forall t \in [0, T].$$

3.2. **Existence of a solution to (11).** We proceed in two steps: first we construct inductively a sequence of smooth global approximate solutions, defined as solutions of a linear system, and then we prove the convergence of this sequence to a solution of the nonlinear system (11) with the required property. Recall that to simplify the presentation we have assumed that $T_0 = +\infty$ and that we focus on the evolution for positive times.

3.2.1. *Construction of the sequence of approximate solutions.* First, we smooth out the data (by convolution for instance) so as to get a sequence $(a_0^n, u_0^n, f^n)_{n \in \mathbb{N}}$ such that $u_0^n \in H^\infty$, $f^n \in \mathcal{C}(\mathbb{R}_+; H^\infty)$, a_0^n and its derivatives at any order are bounded and

$$(31) \quad a_* \leq a_0^n \leq a^*,$$

with in addition

- $a_0^n \rightarrow a_0$ in $B_{\infty,r}^s$,
- $u_0^n \rightarrow u_0$ in $L^2 \cap B_{\infty,r}^s$,
- $f^n \rightarrow f$ in $\mathcal{C}(\mathbb{R}_+; L^2) \cap L^1(\mathbb{R}_+; B_{\infty,r}^s)$.

In order to construct a sequence of smooth approximate solutions, we argue by induction. We first set $a^0 = a_0^0$, $u^0 = u_0^0$ and $\nabla \Pi^0 = 0$.

Now, suppose we have already built a smooth approximate solution $(a^n, u^n, \nabla \Pi^n)$ over $\mathbb{R}_+ \times \mathbb{R}^N$ with a^n satisfying (12). In order to construct the $(n+1)$ -th term of the sequence, we first define a^{n+1} to be the solution of the linear transport equation

$$\partial_t a^{n+1} + u^n \cdot \nabla a^{n+1} = 0$$

with initial datum $a^{n+1}|_{t=0} = a_0^{n+1}$.

Given that u^n is smooth, its flow is smooth too so that $a^{n+1}(t, x) = a_0^{n+1}((\psi_t^n)^{-1}(x))$, where ψ_t^n is the flow at time t . Note that ψ_t^n is a smooth diffeomorphism on the whole \mathbb{R}^N . From this fact, we gather that a^{n+1} is smooth and satisfies (12). Furthermore, by virtue of Proposition 4,

$$(32) \quad \|a^{n+1}(t)\|_{B_{\infty,r}^s} \leq \|a_0^{n+1}\|_{B_{\infty,r}^s} \exp\left(C \int_0^t \|u^n\|_{B_{\infty,r}^s} d\tau\right).$$

Note that the reciprocal function ρ^{n+1} of a^{n+1} satisfies $\rho^{n+1}(t, x) = \rho_0^{n+1}((\psi_t^n)^{-1}(x))$, together with (12) and the equation

$$\partial_t \rho^{n+1} + u^n \cdot \nabla \rho^{n+1} = 0.$$

Hence it also fulfills Inequality (32) up to a change of a_0^{n+1} in ρ_0^{n+1} .

At this point, we define u^{n+1} as the unique smooth solution of the transport equation

$$\begin{cases} \partial_t u^{n+1} + u^n \cdot \nabla u^{n+1} = f^{n+1} - a^{n+1} \nabla \Pi^n \\ u^{n+1}|_{t=0} = u_0^{n+1}. \end{cases}$$

Since the right-hand side belongs to $L_{loc}^1(\mathbb{R}_+; L^2)$, from classical results for transport equation we get that $u^{n+1} \in \mathcal{C}(\mathbb{R}_+; L^2)$. Besides, as $\rho^n = (a^n)^{-1}$ for all n , if we differentiate with respect to time the product $\sqrt{\rho^{n+1}} u^{n+1}$ and take the scalar product with u^{n+1} , we obtain

$$\frac{1}{2} \frac{d}{dt} \left\| \sqrt{\rho^{n+1}} u^{n+1} \right\|_{L^2}^2 = \frac{1}{2} \int \rho^{n+1} |u^{n+1}|^2 \operatorname{div} u^n dx + \int \rho^{n+1} u^{n+1} \cdot f^{n+1} dx - \int \nabla \Pi^n \cdot u^{n+1} dx.$$

Observe that u^n and u^{n+1} need not be divergence free; nevertheless one may control $\|\operatorname{div} u^n\|_{L^\infty}$ with $\|u^n\|_{B_{\infty,r}^s}$. So, from the previous equality, applying Gronwall's Lemma, it is easy to see that

$$(33) \quad \left\| \sqrt{\rho^{n+1}(t)} u^{n+1}(t) \right\|_{L^2} \leq \left\| \sqrt{\rho_0^{n+1}} u_0^{n+1} \right\|_{L^2} + C \int_0^t \left(\|f^{n+1}\|_{L^2} + \|\nabla \Pi^n\|_{L^2} + \|u^n\|_{B_{\infty,r}^s} \right) d\tau.$$

Finally, we have to define the approximate pressure Π^{n+1} . We have already proved that a^{n+1} satisfies the ellipticity hypothesis (12); so we can consider the elliptic equation

$$\operatorname{div} \left(a^{n+1} \nabla \Pi^{n+1} \right) = \operatorname{div} \left(f^{n+1} - u^{n+1} \cdot \nabla \mathcal{P} u^{n+1} \right).$$

As f^{n+1} and u^{n+1} are in $\mathcal{C}(\mathbb{R}_+; H^\infty)$, the classical theory for elliptic equations ensures that the above equation has a unique solution $\nabla \Pi^{n+1}$ in $\mathcal{C}(\mathbb{R}_+; H^\infty)$. In addition, going along the lines of the proof of (27), we get

$$(34) \quad \left\| \nabla \Pi^{n+1} \right\|_{L_t^1(L^2)} \leq C \left(\|f^{n+1}\|_{L_t^1(L^2)} + \int_0^t \|u^{n+1}\|_{B_{\infty,r}^s \cap L^2}^2 d\tau \right).$$

Of course, by embedding, we have $\nabla \Pi^{n+1} \in \mathcal{C}(\mathbb{R}_+; B_{\infty,r}^s)$. Hence, arguing as for proving (28), we get

$$(35) \quad \left\| \nabla \Pi^{n+1} \right\|_{L_t^1(B_{\infty,r}^s)} \leq C \|a^{n+1}\|_{L_t^\infty(B_{\infty,r}^s)} \left(\|f^{n+1}\|_{L_t^1(B_{\infty,r}^s)} + \int_0^t \|u^{n+1}\|_{B_{\infty,r}^s \cap L^2}^2 d\tau \right) \\ + C \left(1 + \|\nabla a^{n+1}\|_{L_t^\infty(B_{\infty,r}^{s-1})}^\gamma \right) \|\nabla \Pi^{n+1}\|_{L_t^1(L^2)}.$$

Note also that the norms of the approximate data that we use in (32), (33), (34) and (35) may be bounded *independently of n* . Therefore, repeating the arguments leading to (30) and to Theorem 1 of [9], one may find some positive time T which may depend on $\|\rho_0\|_{B_{\infty,r}^s}$, $\|u_0\|_{B_{\infty,r}^s \cap L^2}$ and $\|f\|_{L^1([0,T]; B_{\infty,r}^s \cap L^2)}$ but is *independent of n* such that

- $(a^n)_{n \in \mathbb{N}}$ is bounded in $L^\infty([0, T]; B_{\infty,r}^s)$,
- $(u^n)_{n \in \mathbb{N}}$ is bounded in $L^\infty([0, T]; B_{\infty,r}^s \cap L^2)$,
- $(\nabla \Pi^n)_{n \in \mathbb{N}}$ is bounded in $L^1([0, T]; B_{\infty,r}^s) \cap L^\infty([0, T]; L^2)$.

3.2.2. *Convergence of the sequence.* Let us observe that the function $\tilde{a}^n := a^n - a_0^n$ satisfies

$$\begin{cases} \partial_t \tilde{a}^n &= -u^{n-1} \cdot \nabla a^n \\ \tilde{a}^n|_{t=0} &= 0. \end{cases}$$

Because $u^{n-1} \in \mathcal{C}([0, T]; L^2)$ and $\nabla a^n \in \mathcal{C}_b([0, T] \times \mathbb{R}^N)$, it immediately follows that $\tilde{a}^n \in \mathcal{C}^1([0, T]; L^2)$. Now we want to prove that the sequence $(\tilde{a}^n, u^n, \nabla \Pi^n)_{n \in \mathbb{N}}$, built in this way, is a Cauchy sequence in $\mathcal{C}([0, T]; L^2)$. So let us define

$$\begin{aligned} \delta a^n &:= a^{n+1} - a^n, \\ \delta \tilde{a}^n &:= \tilde{a}^{n+1} - \tilde{a}^n = \delta a^n - \delta a_0^n, \\ \delta \rho^n &:= \rho^{n+1} - \rho^n, \\ \delta u^n &:= u^{n+1} - u^n, \\ \delta \Pi^n &:= \Pi^{n+1} - \Pi^n, \\ \delta f^n &:= f^{n+1} - f^n. \end{aligned}$$

Let us emphasize that, by assumption and embedding, we have

- $a_0^n \rightarrow a_0$ in $C^{0,1}$,
- $u_0^n \rightarrow u_0$ in L^2 ,
- $f^n \rightarrow f$ in $L^1([0, T]; L^2)$.

This will be the key to our proof of convergence.

Let us first focus on \tilde{a}^n . By construction, $\delta \tilde{a}^n$ belongs to $\mathcal{C}^1([0, T]; L^2)$ and satisfies the equation

$$\partial_t \delta \tilde{a}^n = -u^n \cdot \nabla \delta \tilde{a}^n - \delta u^{n-1} \cdot \nabla a^n - u^n \cdot \nabla \delta a_0^n$$

from which, taking the scalar product in L^2 with $\delta \tilde{a}^n$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\delta \tilde{a}^n\|_{L^2}^2 = \frac{1}{2} \int (\delta \tilde{a}^n)^2 \operatorname{div} u^n dx - \int \delta u^{n-1} \cdot \nabla a^n \delta \tilde{a}^n dx - \int u^n \cdot \nabla \delta a_0^n \delta \tilde{a}^n dx.$$

So, keeping in mind that $\delta \tilde{a}^n(0) = 0$ and integrating with respect to the time variable one has

$$(36) \quad \|\delta \tilde{a}^n(t)\|_{L^2} \leq \int_0^t \left(\frac{1}{2} \|\operatorname{div} u^n\|_{L^\infty} \|\delta \tilde{a}^n\|_{L^2} + \|\nabla a^n\|_{L^\infty} \|\delta u^{n-1}\|_{L^2} + \|u^n\|_{L^2} \|\nabla \delta a_0^n\|_{L^\infty} \right) d\tau.$$

Equally easily, one can see that the following equality holds true:

$$\rho^{n+1} (\partial_t \delta u^n + u^n \cdot \nabla \delta u^n) + \nabla \delta \Pi^{n-1} = \rho^{n+1} (\delta f^n - \delta u^{n-1} \cdot \nabla u^n - \delta a^n \nabla \Pi^{n-1});$$

taking the scalar product in L^2 with δu^n , integrating by parts, remembering the first equation of (1) at $(n+1)$ -th step, we finally get

$$\begin{aligned} \left\| \sqrt{\rho^{n+1}(t)} \delta u^n(t) \right\|_{L^2} &\leq \int_0^t \|\operatorname{div} u^n\|_{L^\infty} \left\| \sqrt{\rho^{n+1}} \delta u^n \right\|_{L^2} d\tau \\ &+ \int_0^t \left(\|\nabla u^n\|_{L^\infty} \left\| \sqrt{\rho^{n+1}} \delta u^{n-1} \right\|_{L^2} + \left\| \sqrt{\rho^{n+1}} \nabla \Pi^{n-1} \right\|_{L^\infty} \|\delta \tilde{a}^n\|_{L^2} \right. \\ &\quad \left. + \left\| \sqrt{\rho^{n+1}} \nabla \Pi^{n-1} \right\|_{L^2} \|\delta a_0^n\|_{L^\infty} + \left\| \frac{\nabla \delta \Pi^{n-1}}{\sqrt{\rho^{n+1}}} \right\|_{L^2} + \sqrt{\rho^*} \|\delta f^n\|_{L^2} \right) d\tau. \end{aligned}$$

From (36), Gronwall's Lemma and (31), we thus get for some constant C depending only on a_* and a^* ,

$$\begin{aligned} \|(\delta \tilde{a}^n, \delta u^n)(t)\|_{L^2} &\leq C \left(e^{A^n(t)} \|\delta u_0^n\|_{L^2} + \int_0^t e^{A^n(t)-A^n(\tau)} \left(\|(\nabla a^n, \nabla u^n)\|_{L^\infty} \|\delta u^{n-1}\|_{L^2} \right. \right. \\ &\quad \left. \left. + \|\nabla \delta \Pi^{n-1}\|_{L^2} + \|\nabla \Pi^{n-1}\|_{L^2} \|\delta a_0^n\|_{L^\infty} + \|u^n\|_{L^2} \|\nabla \delta a_0^n\|_{L^\infty} + \|\delta f^n\|_{L^2} \right) d\tau, \end{aligned}$$

where we have set

$$A^n(t) := \int_0^t \left(\|\operatorname{div} u^n\|_{L^\infty} + \left\| \sqrt{\rho^{n+1}} \nabla \Pi^{n-1} \right\|_{L^\infty} \right) d\tau.$$

Of course, the uniform a priori estimates of the previous step allow us to control the exponential term for all $t \in [0, T]$ by some constant C_T .

Next, we have to deal with the term $\nabla \delta \Pi^{n-1}$. We notice that it satisfies the elliptic equation

$$-\operatorname{div} \left(a^{n-1} \nabla \delta \Pi^{n-1} \right) = \operatorname{div} \left(-\delta a^{n-1} \nabla \Pi^n - u^{n-1} \cdot \nabla \mathcal{P} \delta u^{n-1} - \delta u^{n-1} \cdot \nabla \mathcal{P} u^n + \delta f^{n-1} \right).$$

Then applying the following algebraic identity

$$\operatorname{div} (v \cdot \nabla w) = \operatorname{div} (w \cdot \nabla v) + \operatorname{div} (v \operatorname{div} w) - \operatorname{div} (w \operatorname{div} v)$$

to $v = u^{n-1}$ and $w = \mathcal{P} \delta u^{n-1}$, and remembering that $\operatorname{div} \mathcal{P} \delta u^{n-1} = 0$, we get

$$\begin{aligned} \operatorname{div} \left(a^{n-1} \nabla \delta \Pi^{n-1} \right) &= \operatorname{div} \left(\mathcal{P} \delta u^{n-1} \operatorname{div} u^{n-1} - \mathcal{P} \delta u^{n-1} \cdot \nabla u^{n-1} - \delta u^{n-1} \cdot \nabla \mathcal{P} u^n \right. \\ &\quad \left. - \delta a^{n-1} \nabla \Pi^n + \delta f^{n-1} \right). \end{aligned}$$

Therefore, from Lemma 2 and the fact that $\|\mathcal{P}\|_{\mathcal{L}(L^2; L^2)} = 1$, one immediately has the following inequality:

$$(37) \quad a_* \left\| \nabla \delta \Pi^{n-1} \right\|_{L^2} \leq \left\| \tilde{\delta a}^{n-1} \right\|_{L^2} \|\nabla \Pi^n\|_{L^\infty} + \left\| \delta a_0^{n-1} \right\|_{L^\infty} \|\nabla \Pi^n\|_{L^2} + \|\delta f^{n-1}\|_{L^2} \\ + \left\| \delta u^{n-1} \right\|_{L^2} \left(\left\| \operatorname{div} u^{n-1} \right\|_{L^\infty} + \left\| \nabla u^{n-1} \right\|_{L^\infty} + \|\nabla \mathcal{P} u^n\|_{L^\infty} \right).$$

Due to a priori estimates, we finally obtain, for all $t \in [0, T]$,

$$\begin{aligned} \|(\tilde{\delta a}^n, \delta u^n)(t)\|_{L^2} &\leq C_T \left(\|\delta u_0^n\|_{L^2} + \int_0^t \left(\|(\tilde{\delta a}^{n-1}, \delta u^{n-1})\|_{L^2} \right. \right. \\ &\quad \left. \left. + \left\| \nabla \delta \Pi^{n-1} \right\|_{L^2} + \|\delta a_0^n\|_{C^{0,1}} + \|\delta f^n\|_{L^2} \right) d\tau \right) \\ \left\| \nabla \delta \Pi^{n-1} \right\|_{L^2} &\leq C_T \left(\left\| \tilde{\delta a}^{n-1} \right\|_{L^2} + \left\| \delta u^{n-1} \right\|_{L^2} + \|\delta a_0^{n-1}\|_{L^\infty} + \|\delta f^{n-1}\|_{L^2} \right); \end{aligned}$$

so, plugging the second inequality in the first one, we find out that for all $t \in [0, T]$,

$$(38) \quad \|(\tilde{\delta a}^n, \delta u^n)(t)\|_{L^2} \leq \varepsilon_n + C_T \int_0^t \|(\tilde{\delta a}^{n-1}, \delta u^{n-1})\|_{L^2} d\tau$$

with

$$\varepsilon_n := C_T \left(\|\delta u_0^n\|_{L^2} + \int_0^T \left(\|\delta f^{n-1}\|_{L^2} + \|\delta f^n\|_{L^2} + \|\delta a_0^{n-1}\|_{L^\infty} + \|\nabla \delta a_0^{n-1}\|_{L^\infty} \right) dt \right).$$

Now, we have

$$\sum_n \varepsilon_n < \infty.$$

From this and (38), it is easy to conclude that

$$\sum_n \sup_{t \in [0, T]} \left(\|\tilde{\delta a}^n(t)\|_{L^2} + \|\delta u^n(t)\|_{L^2} \right) < \infty.$$

In other words, $(\tilde{a}^n)_{n \in \mathbb{N}}$ and $(u^n)_{n \in \mathbb{N}}$ are Cauchy sequences in $\mathcal{C}([0, T]; L^2)$; therefore they converge to some functions \tilde{a} , $u \in \mathcal{C}([0, T]; L^2)$. In the same way, it is clear that $(\nabla \Pi^n)_{n \in \mathbb{N}}$ converges to some $\nabla \Pi \in \mathcal{C}([0, T]; L^2)$.

Defining $a := \tilde{a} + a_0$, it remains to show that a , u and $\nabla \Pi$ are indeed solutions of the initial system. We already know that a , u and $\nabla \Pi \in \mathcal{C}([0, T]; L^2)$. In addition,

- thanks to Fatou's property in Besov spaces, as $(a^n)_{n \in \mathbb{N}}$ is bounded in $L^\infty([0, T]; B_{\infty, r}^s)$, we obtain that $a \in L^\infty([0, T]; B_{\infty, r}^s)$ and satisfies (31);

- in the same way, $u \in L^\infty([0, T]; B_{\infty, r}^s)$ because also $(u^n)_{n \in \mathbb{N}}$ is bounded in the same space;
- finally, $\nabla \Pi \in L^1([0, T]; B_{\infty, r}^s)$ because the sequence $(\nabla \Pi^n)_{n \in \mathbb{N}}$ is bounded in the same functional space.

By interpolation we get that the sequences converge strongly to the solutions in every intermediate space between $\mathcal{C}([0, T]; L^2)$ and $\mathcal{C}([0, T]; B_{\infty, r}^s)$, that is enough to pass to the limit in the equations satisfied by $(a^n, u^n, \nabla \Pi^n)$. So, $(a, u, \nabla \Pi)$ satisfies System (11).

Finally, continuity properties of the solutions with respect to the time variable can be recovered from the equations satisfied by them, using classical results for transport equations in Besov spaces (see Proposition 4).

3.3. Uniqueness of the solution. Uniqueness of the solution to System (1) is a straightforward consequence of the following stability result, the proof of which can be found in [9].

Proposition 5. *Let $(\rho_1, u_1, \nabla \Pi_1)$ and $(\rho_2, u_2, \nabla \Pi_2)$ satisfy System (1) with external forces f_1 and f_2 , respectively. Suppose that ρ_1 and ρ_2 both satisfy (12). Assume also that:*

- $\delta \rho := \rho_2 - \rho_1$ and $\delta u := u_2 - u_1$ both belong to $\mathcal{C}^1([0, T]; L^2)$,
- $\delta f := f_2 - f_1 \in \mathcal{C}([0, T]; L^2)$,
- $\nabla \rho_1, \nabla u_1$ and $\nabla \Pi_1$ belong to $L^1([0, T]; L^\infty)$.

Then for all $t \in [0, T]$ we have

$$e^{-A(t)} (\|\delta \rho(t)\|_{L^2} + \|(\sqrt{\rho_2} \delta u)(t)\|_{L^2}) \leq \|\delta \rho(0)\|_{L^2} + \|(\sqrt{\rho_2} \delta u)(0)\|_{L^2} + \int_0^t e^{-A(\tau)} \|(\sqrt{\rho_2} \delta f)\|_{L^2} d\tau$$

with

$$A(t) := \int_0^t \left(\left\| \frac{\nabla \rho_1}{\sqrt{\rho_2}} \right\|_{L^\infty} + \left\| \frac{\nabla \Pi_1}{\rho_1 \sqrt{\rho_2}} \right\|_{L^\infty} + \|\nabla u_1\|_{L^\infty} \right) d\tau.$$

Proof of uniqueness in Theorem 1. Let us suppose that there exist two solutions $(\rho_1, u_1, \nabla \Pi_1)$ and $(\rho_2, u_2, \nabla \Pi_2)$ to System (1) corresponding to the same data and satisfying the hypotheses of Theorem 1. Then, as one can easily verify, these solutions satisfy the assumptions of Proposition 5. For instance, that $\delta \rho \in \mathcal{C}^1([0, T]; L^2)$ is an immediate consequence of the fact that, for $i = 1, 2$, the velocity field u_i is in $\mathcal{C}([0, T]; L^2)$ and $\nabla \rho_i$ is in $\mathcal{C}([0, T]; L^\infty)$, so that $\partial_t \rho_i \in \mathcal{C}([0, T]; L^2)$.

So, Proposition 5 implies that $(\rho_1, u_1, \nabla \Pi_1) \equiv (\rho_2, u_2, \nabla \Pi_2)$. \square

3.4. Proof of the continuation criterion. Now, we want to prove the continuation criterion for the solution to (1). We proceed in two steps. As usual, we will suppose Condition (C) to be satisfied with $p = \infty$. The first step of the proof is given by the following lemma.

Lemma 4. *Let $(\rho, u, \nabla \Pi)$ be a solution of System (1) on $[0, T^*[\times \mathbb{R}^N$ such that²*

- $u \in \mathcal{C}([0, T^*]; B_{\infty, r}^s) \cap \mathcal{C}^1([0, T^*]; L^2)$,
- $\rho \in \mathcal{C}([0, T^*]; B_{\infty, r}^s)$ and satisfies (12).

Suppose also that Condition (4) holds and that T^* is finite. Then

$$\sup_{t \in [0, T^*[} \left(\|u(t)\|_{B_{\infty, r}^s \cap L^2} + \|\rho(t)\|_{B_{\infty, r}^s} \right) + \int_0^{T^*} \|\nabla \Pi\|_{B_{\infty, r}^s} dt < \infty.$$

Proof of Lemma 4. It is only a matter of repeating the a priori estimates of the previous section, but in a more accurate way. Note that $a := 1/\rho$ satisfies the same hypothesis as ρ , so we will work without distinction with these two quantities, according to what is more convenient to us, and set $q = \rho$ or a . Recall that

$$\partial_t q + u \cdot \nabla q = 0.$$

Hence, applying operator Δ_j yields

$$\partial_t \Delta_j q + u \cdot \nabla \Delta_j q = [u \cdot \nabla, \Delta_j] q$$

²with the usual convention that continuity in time is weak if $r = \infty$.

whence, for all $t \in [0, T^*[,$

$$(39) \quad 2^{js} \|\Delta_j q(t)\|_{L^\infty} \leq 2^{js} \|\Delta_j q_0\|_{L^\infty} + \int_0^t 2^{js} \|[u \cdot \nabla, \Delta_j]q\|_{L^\infty} d\tau.$$

Now, Lemma 2.100 in [1] ensures that

$$\left\| (2^{js} \|[u \cdot \nabla, \Delta_j]q\|_{L^\infty})_j \right\|_{\ell^r} \leq C (\|\nabla u\|_{L^\infty} \|q\|_{B_{\infty,r}^s} + \|\nabla q\|_{L^\infty} \|\nabla u\|_{B_{\infty,r}^{s-1}}).$$

Hence, performing an ℓ^r summation in (39), we get

$$(40) \quad \|q(t)\|_{B_{\infty,r}^s} \leq \|q_0\|_{B_{\infty,r}^s} + C \int_0^t (\|\nabla u\|_{L^\infty} \|q\|_{B_{\infty,r}^s} + \|\nabla q\|_{L^\infty} \|u\|_{B_{\infty,r}^s}) d\tau.$$

As regards the velocity field, we have according to (14),

$$\|u(t)\|_{L^2} \leq C \left(\|u_0\|_{L^2} + \int_0^t \|f\|_{L^2} d\tau \right),$$

and the last part of Proposition 4 guarantees that

$$\begin{aligned} \|u(t)\|_{B_{\infty,r}^s} &\leq \exp \left(C \int_0^t \|\nabla u\|_{L^\infty} d\tau \right) \\ &\quad \times \left(\|u_0\|_{B_{\infty,r}^s} + \int_0^t e^{-C \int_0^\tau \|\nabla u\|_{L^\infty} d\tau'} (\|f\|_{B_{\infty,r}^s} + \|a \nabla \Pi\|_{B_{\infty,r}^s}) d\tau \right). \end{aligned}$$

Bounding the last term according to Corollary 3, we thus get

$$\begin{aligned} \|u(t)\|_{B_{\infty,r}^s} &\leq \exp \left(C \int_0^t \|\nabla u\|_{L^\infty} d\tau \right) \\ &\quad \times \left(\|u_0\|_{B_{\infty,r}^s} + \int_0^t e^{-C \int_0^\tau \|\nabla u\|_{L^\infty} d\tau'} (\|f\|_{B_{\infty,r}^s} + a^* \|\nabla \Pi\|_{B_{\infty,r}^s} + \|\nabla a\|_{B_{\infty,r}^{s-1}} \|\nabla \Pi\|_{L^\infty}) d\tau \right). \end{aligned}$$

As regards the pressure term, we have

$$\begin{aligned} \|\nabla \Pi\|_{L^2} &\leq C (\|f\|_{L^2} + \|u\|_{L^2} \|\nabla u\|_{L^\infty}) \\ \|\nabla \Pi\|_{B_{\infty,r}^s} &\leq C \left(\|\nabla \Pi\|_{L^2} + \|\nabla a \cdot \nabla \Pi\|_{B_{\infty,r}^{s-1}} + \left\| \frac{1}{a} \operatorname{div} (f - u \cdot \nabla u) \right\|_{B_{\infty,r}^{s-1}} \right). \end{aligned}$$

Note that Bony's decomposition combined with the fact that $\operatorname{div} u = 0$ ensures that

$$\|\operatorname{div} (u \cdot \nabla u)\|_{B_{\infty,r}^{s-1}} \leq C \|\nabla u\|_{L^\infty} \|u\|_{B_{\infty,r}^s}.$$

In addition, *under the assumption that $s > 1$* , Corollary 3 implies that

$$(41) \quad \|\nabla a \cdot \nabla \Pi\|_{B_{\infty,r}^{s-1}} \leq C (\|\nabla a\|_{L^\infty} \|\nabla \Pi\|_{B_{\infty,r}^{s-1}} + \|\nabla a\|_{B_{\infty,r}^{s-1}} \|\nabla \Pi\|_{L^\infty}).$$

So finally

$$\begin{aligned} \|\nabla \Pi\|_{B_{\infty,r}^s} &\leq C \left(\|\nabla \Pi\|_{L^2} + \|\nabla a\|_{L^\infty} \|\nabla \Pi\|_{B_{\infty,r}^{s-1}} + \|\nabla a\|_{B_{\infty,r}^{s-1}} \|\nabla \Pi\|_{L^\infty} \right. \\ &\quad \left. + \|a\|_{B_{\infty,r}^s} (\|f\|_{B_{\infty,r}^s} + \|\nabla u\|_{L^\infty} \|u\|_{B_{\infty,r}^s}) \right). \end{aligned}$$

Putting together all these estimates and applying Gronwall's Lemma, we obtain if $s > 1$,

$$\begin{aligned} \|\nabla a\|_{B_{\infty,r}^{s-1}} + \|u(t)\|_{B_{\infty,r}^s \cap L^2} &\leq \exp \left(C \int_0^t \|(\nabla a, \nabla u, \nabla \Pi)\|_{L^\infty} d\tau \right) \left(\|\nabla a_0\|_{B_{\infty,r}^{s-1}} + \right. \\ &\quad \left. \|u_0\|_{B_{\infty,r}^s \cap L^2} + \|f\|_{B_{\infty,r}^s \cap L^2} + \int_0^t \|\nabla a\|_{L^\infty} \|\nabla \Pi\|_{B_{\infty,r}^{s-1}} d\tau \right), \end{aligned}$$

where the constant C depends only on s , a_* , a^* and N .

Now, the equation for ∇a and Gronwall inequality immediately ensure that

$$(42) \quad \|\nabla a(t)\|_{L^\infty} \leq \|\nabla a_0\|_{L^\infty} \exp \left(\int_0^t \|\nabla u\|_{L^\infty} d\tau \right),$$

which, thanks to Hypothesis (4) implies that ∇a is bounded in time with values in L^∞ .

Moreover, by hypothesis $\nabla \Pi \in L^1([0, T^*]; B_{\infty, r}^{s-1})$ and $\nabla u \in L^1([0, T^*]; L^\infty)$; at this point, keeping in mind the embedding $B_{\infty, r}^{s-1} \hookrightarrow L^\infty$, the previous inequality gives us the thesis of the lemma in the case $s > 1$.

In the endpoint case $s = r = 1$, Inequality (41) fails. In order to complete the proof of the lemma, we will have to take advantage of the vorticity equation associated to (1). This is postponed to the next section. \square

The second lemma, which will enable us to complete the proof of Theorem 2 reads:

Lemma 5. *Let $(\rho, u, \nabla \Pi)$ be the solution of System (1) such that³*

- $\rho \in \mathcal{C}([0, T^*]; B_{\infty, r}^s)$ and (12);
- $u \in \mathcal{C}([0, T^*]; B_{\infty, r}^s) \cap \mathcal{C}^1([0, T^*]; L^2)$;
- $\nabla \Pi \in \mathcal{C}([0, T^*]; L^2) \cap L^1([0, T^*]; B_{\infty, r}^s)$.

Moreover, suppose that

$$\|u\|_{L_{T^*}^\infty(B_{\infty, r}^s \cap L^2)} + \|\nabla a\|_{L_{T^*}^\infty(B_{\infty, r}^{s-1})} < \infty.$$

Then $(\rho, u, \nabla \Pi)$ can be continued beyond time T^* into a solution of (1) with the same regularity.

Proof of Lemma 5. From the proof of Theorem 1 we know that there exists a time ε , depending only on ρ^* , N , s , $\|u\|_{L_{T^*}^\infty(B_{\infty, r}^s \cap L^2)}$, $\|\nabla a\|_{L_{T^*}^\infty(B_{\infty, r}^{s-1})}$ and on the norm of the data such that, for all $\tilde{T} < T$, Euler system with data $(\rho(\tilde{T}), u(\tilde{T}), f(\tilde{T} + \cdot))$ has a unique solution until time ε .

Now, taking for example $\tilde{T} = T - \varepsilon/2$, we thus obtain a solution, which is the continuation of the initial one, $(\rho, u, \nabla \Pi)$, until time $T + \varepsilon/2$. \square

Let us complete the proof of Theorem 2. The first part is a straightforward consequence of these two lemmas. Indeed: Lemma 4 ensures that $\|u\|_{L_{T^*}^\infty(B_{\infty, r}^s \cap L^2)}$ and $\|\nabla a\|_{L_{T^*}^\infty(B_{\infty, r}^{s-1})}$ are finite. As for the last claim (the Beale-Kato-Majda type continuation criterion), it is a classical consequence of the well-known logarithmic interpolation inequality (see e.g. [1])

$$\|\nabla u\|_{L^\infty} \leq C \left(\|u\|_{L^2} + \|\Omega\|_{L^\infty} \log \left(e + \frac{\|\Omega\|_{B_{\infty, r}^{s-1}}}{\|\Omega\|_{L^\infty}} \right) \right).$$

So Theorem 2 is now completely proved, up to the endpoint case $s = r = 1$. \square

4. THE VORTICITY EQUATION AND APPLICATIONS

This section is devoted to the proof of the blow-up criterion in the endpoint case $s = r = 1$, and of Theorem 3. Both results rely on the vorticity equation associated to System (1). As done in Section 3, we shall restrict ourselves to the evolution for positive times and make the usual convention as regards time continuity, if $r < \infty$.

4.1. On the vorticity. As in all this section the vorticity will play a fundamental role, let us spend some words about it. Given a vector-field u , we set ∇u its Jacobian matrix and ${}^t\nabla u$ the transposed matrix of ∇u . We define the vorticity associated to u by

$$\Omega := \nabla u - {}^t\nabla u.$$

Recall that, in dimension $N = 2$, Ω can be identified with the scalar function $\omega = \partial_1 u^2 - \partial_2 u^1$, while for $N = 3$ with the vector-field $\omega = \nabla \times u$.

³with the usual convention that continuity in time is weak if $r = \infty$.

It is obvious that, for all $q \in [1, \infty]$, if $\nabla u \in L^q$, then also $\Omega \in L^q$. Conversely, if u is divergence-free then for all $1 \leq i \leq N$ we have $\Delta u^i = \sum_{j=1}^N \partial_j \Omega_{ij}$, and so, formally,

$$\nabla u^i = -\nabla(-\Delta)^{-1} \sum_{j=1}^N \partial_j \Omega_{ij}.$$

As the symbol of the operator $-\partial_i(-\Delta)^{-1} \partial_j$ is $\sigma(\xi) = \xi_i \xi_j / |\xi|^2$, the classical Calderon-Zygmund Theorem ensures that⁴ for all $q \in]1, \infty[$ if $\Omega \in L^q$, then $\nabla u \in L^q$ and

$$(43) \quad \|\nabla u\|_{L^q} \leq C \|\Omega\|_{L^q}.$$

The above relation also implies that

$$u = \Delta_{-1} u - (\text{Id} - \Delta_{-1})(-\Delta)^{-1} \sum_j \partial_j \Omega_{ij}.$$

Hence combining Bernstein's inequality and Proposition 1, we gather that

$$(44) \quad \|u\|_{B_{\infty,1}^1} \leq C(\|u\|_{L^q} + \|\Omega\|_{B_{\infty,1}^0}) \quad \text{for all } q \in [1, \infty].$$

From now on, let us assume that Ω is the vorticity associated to some solution $(\rho, u, \nabla \Pi)$ of (1), defined on $[0, T] \times \mathbb{R}^N$. From the velocity equation, we gather that Ω satisfies the following transport-like equation:

$$(45) \quad \partial_t \Omega + u \cdot \nabla \Omega + \Omega \cdot \nabla u + {}^t \nabla u \cdot \Omega + \nabla \left(\frac{1}{\rho} \right) \wedge \nabla \Pi = F$$

where $F_{ij} := \partial_j f^i - \partial_i f^j$ and, for two vector fields v and w , we have set $v \wedge w$ to be the skew-symmetric matrix with components

$$(v \wedge w)_{ij} = v^j w^i - v^i w^j.$$

Using classical L^q estimates for transport equations and taking advantage of Gronwall's Lemma, from (45) we immediately get

$$(46) \quad \|\Omega(t)\|_{L^q} \leq \exp\left(\int_0^t \|\nabla u\|_{L^\infty} d\tau\right) \times \left(\|\Omega(0)\|_{L^q} + \int_0^t e^{-\int_0^\tau \|\nabla u\|_{L^\infty} d\tau'} \left(\|F\|_{L^q} + \left\| \frac{1}{\rho^2} \nabla \rho \wedge \nabla \Pi \right\|_{L^q} \right) d\tau\right).$$

Let us notice that, in the case of space dimension $N = 2$, equation (45) becomes

$$\partial_t \omega + u \cdot \nabla \omega + \nabla \left(\frac{1}{\rho} \right) \wedge \nabla \Pi = F,$$

so that one obtains the same estimate as before, but without the exponential growth:

$$\|\omega(t)\|_{L^q} \leq \|\omega(0)\|_{L^q} + \int_0^t \left(\|F\|_{L^q} + \left\| \frac{1}{\rho^2} \nabla \rho \wedge \nabla \Pi \right\|_{L^q} \right) d\tau.$$

Therefore, the two-dimensional case is in a certain sense better. We shall take advantage of that in Section 5. As concerns the results of this section, the proof will not depend on the dimension. So for the time being we assume that the dimension N is any integer greater than or equal to 2.

⁴This time the extreme values are not included.

4.2. Proof of Theorem 2 in the limit case $s = r = 1$. We just have to modify the proof of Lemma 4. From the vorticity equation (45) and Proposition 4 (recall that $\operatorname{div} u = 0$), we readily get

$$(47) \quad \|\Omega(t)\|_{B_{\infty,1}^0} \leq \exp\left(C \int_0^t \|\nabla u\|_{L^\infty} d\tau\right) \\ \times \left(\|\Omega_0\|_{B_{\infty,1}^0} + \int_0^t \|F\|_{B_{\infty,1}^0} d\tau + \int_0^t (\|\nabla a \wedge \nabla \Pi\|_{B_{\infty,1}^0} + \|\Omega \cdot \nabla u + {}^t\nabla u \cdot \Omega\|_{B_{\infty,1}^0}) d\tau \right).$$

We claim that

$$(48) \quad \|\nabla a \wedge \nabla \Pi\|_{B_{\infty,1}^0} \leq C(\|\nabla a\|_{L^\infty} \|\nabla \Pi\|_{B_{\infty,1}^0} + \|\nabla \Pi\|_{L^\infty} \|a\|_{B_{\infty,1}^1}),$$

$$(49) \quad \|\Omega \cdot \nabla u + {}^t\nabla u \cdot \Omega\|_{B_{\infty,1}^0} \leq C\|\nabla u\|_{L^\infty} \|u\|_{B_{\infty,1}^1}.$$

Both inequalities rely on Bony's decomposition (6) and algebraic cancellations. Indeed, we observe that

$$\partial_i a \partial_j \Pi - \partial_j a \partial_i \Pi = T_{\partial_i a} \partial_j \Pi - T_{\partial_j a} \partial_i \Pi + T_{\partial_j \Pi} \partial_i a - T_{\partial_i \Pi} \partial_j a + \partial_i R(a, \partial_j \Pi) - \partial_j R(a, \partial_i \Pi).$$

Applying Proposition 2 thus yields (48).

Next, we notice that, as $\operatorname{div} u = 0$,

$$(\Omega \cdot \nabla u + {}^t\nabla u \cdot \Omega)_{ij} = \sum_k (\partial_i u^k \partial_k u^j - \partial_j u^k \partial_k u^i), \\ = \sum_k (\partial_k (u^j \partial_i u^k) - \partial_k (u^i \partial_j u^k)).$$

Therefore,

$$(\Omega \cdot \nabla u + {}^t\nabla u \cdot \Omega)_{ij} \\ = \sum_k \left(T_{\partial_i u^k} \partial_k u^j - T_{\partial_j u^k} \partial_k u^i + T_{\partial_k u^j} \partial_i u^k - T_{\partial_k u^i} \partial_j u^k + \partial_k R(u^j, \partial_i u^k) - \partial_k R(u^i, \partial_j u^k) \right).$$

Hence Proposition 2 implies (49).

It is now easy to complete the proof of Lemma 4 in the limit case. Indeed, plugging (48) and (49) in (47), using the energy inequality (14) and Inequality (44) with $q = 2$, we easily get

$$\|u(t)\|_{B_{\infty,1}^1 \cap L^2} \leq C \exp\left(C \int_0^t \|\nabla u\|_{L^\infty} d\tau\right) \\ \times \left(\|u_0\|_{B_{\infty,1}^1 \cap L^2} + \int_0^t \|f\|_{B_{\infty,1}^1 \cap L^2} d\tau + \int_0^t (\|\nabla a\|_{L^\infty} \|\nabla \Pi\|_{B_{\infty,1}^0} + \|\nabla \Pi\|_{L^\infty} \|a\|_{B_{\infty,1}^1}) d\tau \right).$$

Hence, adding up Inequality (40) and using Gronwall's inequality, we end up with

$$X(t) \leq C \exp\left(C \int_0^t \|\nabla u, \nabla a, \nabla \Pi\|_{L^\infty} d\tau\right) \left(X(0) + \int_0^t (\|f\|_{B_{\infty,1}^1 \cap L^2} + \|\nabla a\|_{L^\infty} \|\nabla \Pi\|_{B_{\infty,1}^0}) d\tau \right)$$

with $X(t) := \|a(t)\|_{B_{\infty,1}^1} + \|u(t)\|_{B_{\infty,1}^1 \cap L^2}$.

Taking advantage of (42) completes the proof of Lemma 4 in the limit case. \square

4.3. Proof of Theorem 3. We first prove a priori estimates, and then we will get from them existence and uniqueness of the solution. In fact, it will turn out to be possible to apply Theorem 1 after performing a suitable cut-off on the initial velocity field and thus to work directly on System (1), without passing through the equivalence with (11) or with a sequence of approximate linear systems.

4.3.1. *A priori estimates.* As in the previous section, remembering also Remark 3, the following estimates hold true:

$$(50) \quad \|\nabla \rho(t)\|_{B_{\infty,r}^{s-1}} \leq \|\nabla \rho_0\|_{B_{\infty,r}^{s-1}} \exp\left(C \int_0^t \|u\|_{B_{\infty,r}^s} d\tau\right)$$

$$(51) \quad \begin{aligned} \|u(t)\|_{B_{\infty,r}^s} &\leq \exp\left(C \int_0^t \|u\|_{B_{\infty,r}^s} d\tau\right) \cdot \left(\|u_0\|_{B_{\infty,r}^s} + \right. \\ &\quad \left. + \int_0^t e^{-C \int_0^\tau \|u\|_{B_{\infty,r}^s} d\tau'} \|\rho\|_{B_{\infty,r}^s} \|\nabla \Pi\|_{B_{\infty,r}^s} d\tau\right). \end{aligned}$$

Moreover, from the transport equation satisfied by the velocity field, we easily gather that

$$\|u(t)\|_{L^4} \leq \|u_0\|_{L^4} + \int_0^t \left\| \frac{\nabla \Pi}{\rho} \right\|_{L^4} d\tau.$$

Therefore, using interpolation in Lebesgue spaces and embedding (see Corollary 2),

$$(52) \quad \begin{aligned} \|u(t)\|_{L^4} &\leq \|u_0\|_{L^4} + \frac{1}{\rho_*} \int_0^t \|\nabla \Pi\|_{L^\infty}^{\frac{1}{2}} \|\nabla \Pi\|_{L^2}^{\frac{1}{2}} d\tau \\ &\leq \|u_0\|_{L^4} + \frac{C}{\rho_*} \int_0^t \|\nabla \Pi\|_{B_{\infty,r}^s \cap L^2} d\tau. \end{aligned}$$

In order to bound the vorticity in L^4 , one may use that

$$\begin{aligned} \left\| \frac{1}{\rho^2} \nabla \rho \wedge \nabla \Pi \right\|_{L^4} &\leq \frac{1}{(\rho_*)^2} \|\nabla \rho \wedge \nabla \Pi\|_{L^4} \\ &\leq \frac{1}{(\rho_*)^2} \|\nabla \rho\|_{L^\infty} \|\nabla \Pi\|_{L^4} \\ &\leq \frac{C}{(\rho_*)^2} \|\nabla \rho\|_{B_{\infty,r}^{s-1}} \|\nabla \Pi\|_{B_{\infty,r}^s \cap L^2}. \end{aligned}$$

From this and (46), we thus get

$$(53) \quad \begin{aligned} \|\Omega(t)\|_{L^4} &\leq \exp\left(\int_0^t \|\nabla u\|_{B_{\infty,r}^{s-1}} d\tau\right) \\ &\quad \times \left(\|\Omega_0\|_{L^4} + \frac{C}{(\rho_*)^2} \int_0^t e^{-\int_0^\tau \|\nabla u\|_{B_{\infty,r}^{s-1}} d\tau'} \|\nabla \rho\|_{B_{\infty,r}^{s-1}} \|\nabla \Pi\|_{B_{\infty,r}^s \cap L^2} d\tau \right). \end{aligned}$$

Now, in order to close the estimates, we need to control the pressure term. Its Besov norm can be bounded as in Section 3, up to a change of $\|u\|_{L^2}$ into $\|u\|_{L^4}$; indeed it is clear that in Inequality (25) the L^2 norm of u may be replaced by any L^q norm with $q < \infty$. As a consequence, combining the (modified) inequality (26) and (21) yields

$$(54) \quad \|\nabla \Pi\|_{L_t^1(B_{\infty,r}^s)} \leq C \left(\left(1 + \|\nabla a\|_{L_t^\infty(B_{\infty,r}^{s-1})}^\gamma\right) \|\nabla \Pi\|_{L_t^1(L^2)} + \|\rho\|_{L_t^\infty(B_{\infty,r}^s)} \int_0^t \|u\|_{B_{\infty,r}^s \cap L^4}^2 d\tau \right).$$

In order to bound the L^2 norm of $\nabla \Pi$, we take the divergence of the second equation of System (1). We obtain

$$-\operatorname{div} \left(\frac{\nabla \Pi}{\rho} \right) = \operatorname{div} (u \cdot \nabla u),$$

from which, applying elliptic estimates of Lemma 2 and

$$(55) \quad \|\nabla u\|_{L^4} \leq C \|\Omega\|_{L^4},$$

we get

$$(56) \quad \frac{1}{\rho_*} \|\nabla \Pi\|_{L^2} \leq \|u \cdot \nabla u\|_{L^2} \leq \|u\|_{L^4} \|\nabla u\|_{L^4} \leq C \|u\|_{L^4} \|\Omega\|_{L^4}.$$

We claim that putting together inequalities (50), (51), (52), (55), (54), (53) and (56) enables us to close the estimates on some nontrivial time interval $[0, T]$ depending only on the norm of the data.

In effect, assuming that T has been chosen so that Inequality (29) is satisfied, we get from the above inequalities

$$\begin{aligned} \|u(t)\|_{B_{\infty,r}^s} &\leq 2\|u_0\|_{B_{\infty,r}^s} + C_0\|\nabla\Pi\|_{L_t^1(B_{\infty,r}^s)}, \\ \|\nabla\Pi\|_{L_t^1(B_{\infty,r}^s)} &\leq C_0\left(\int_0^t (\|u\|_{L^4}\|\Omega\|_{L^4} + \|u\|_{B_{\infty,r}^s\cap L^4}^2) d\tau\right), \\ \|u(t)\|_{L^4} &\leq \|u_0\|_{L^4} + C_0\|\nabla\Pi\|_{L_t^1(B_{\infty,r}^s)} + C_0\int_0^t \|u\|_{L^4}\|\Omega\|_{L^4} d\tau, \\ \|\Omega(t)\|_{L^4} &\leq 2\|\Omega_0\|_{L^4} + C_0\|\nabla\Pi\|_{L_t^1(B_{\infty,r}^s)}, \end{aligned}$$

where the constant C_0 depends on s , ρ_* , ρ^* , N and $\|\rho_0\|_{B_{\infty,1}^1}$.

Therefore, applying Gronwall lemma and assuming that T has been chosen so that (in addition to (29)) we have

$$\int_0^T \|u\|_{W^{1,4}} d\tau \leq c$$

where c is a small enough constant depending only on C_0 , it is easy to close the estimates.

Remark 4. *Exhibiting an L^2 estimate for $\nabla\Pi$ even though u is not in L^2 is the key to the proof. This has been obtained in (56). Note however that we have some freedom there. In fact, one may rather assume that $u_0 \in L^p$ and $\nabla u_0 \in L^q$, with p and q in $]2, \infty[$ such that $1/p + 1/q \geq 1/2$ and get a statement similar to that of Theorem 3 under these two assumptions. The details are left to the reader.*

4.3.2. Existence of a solution. We want to take advantage of the existence theory provided by Theorem 1. However, under the assumptions of Theorem 3, the initial velocity does not belong to L^2 . To overcome this, we shall introduce a sequence of truncated initial velocities. Then Theorem 1 will enable us to solve System (1) with these modified data and the previous part will provide uniform estimates in the right functional spaces on a small enough (fixed) time interval. Finally, convergence will be proved by an energy method similar to that we used for Theorem 1.

First step: construction of the sequence of approximate solutions. Take any $\Phi \in C_0^\infty(\mathbb{R}_x^N)$ with $\Phi \equiv 1$ on a neighborhood of the origin, and set $\Phi_n(x) = \Phi(x/n)$. Then let us define $u_0^n := \Phi_n u_0$ for all $n \in \mathbb{N}$.

Given that u_0^n is continuous and compactly supported, it obviously belongs to L^2 . Of course, we still have $u_0^n \in B_{\infty,r}^s \cap W^{1,4} \cap L^2$, so we fall back into hypothesis of Theorem 1. From it, we get the existence of some time T_n and of a solution $(\rho^n, u^n, \nabla\Pi^n)$ to (1) with data $(\rho_0, u_0^n, 0)$ such that $\rho^n \in \mathcal{C}([0, T_n]; B_{\infty,r}^s)$, $u^n \in \mathcal{C}^1([0, T_n]; L^2) \cap \mathcal{C}([0, T_n]; B_{\infty,r}^s)$ and $\nabla\Pi^n \in \mathcal{C}([0, T_n]; L^2) \cap L^1([0, T_n]; B_{\infty,r}^s)$. From (55), the vorticity equation and the velocity equation, it is easy to see that, in addition, $u^n \in \mathcal{C}([0, T_n]; W^{1,4})$.

Finally, as the norm of u_0^n in $W^{1,4} \cap B_{\infty,r}^s$ may be bounded independently of n , the a priori estimates that have been performed in the previous paragraph ensure that one may find some positive lower bound T for T_n such that $(\rho^n, u^n, \nabla\Pi^n)$ satisfies bounds independent of n on $[0, T]$ in the desired functional spaces.

Second step: convergence of the sequence. As done in the previous section, we define $\tilde{\rho}^n = \rho^n - \rho_0$, and then

$$\begin{aligned} \delta\rho^n &:= \tilde{\rho}^{n+1} - \tilde{\rho}^n, \\ \delta u^n &:= u^{n+1} - u^n, \\ \delta\Pi^n &:= \Pi^{n+1} - \Pi^n. \end{aligned}$$

Resorting to the same type of computations as in the previous section (it is actually easier as, now, $\operatorname{div} u^n = 0$ for all n), we can prove that $(\tilde{\rho}^n, u^n, \nabla \Pi^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0, T]; L^2)$. Hence it converges to some $(\tilde{\rho}, u, \nabla \Pi)$ which belongs to the same space.

Now, defining $\rho := \rho_0 + \tilde{\rho}$, bearing in mind the uniform estimates of the previous step, and using the Fatou property, we easily conclude that

- $\rho \in L^\infty([0, T]; B_{\infty, r}^s)$ and $\rho_* \leq \rho \leq \rho^*$;
- $u \in L^\infty([0, T]; B_{\infty, r}^s) \cap L^\infty([0, T]; W^{1,4})$;
- $\nabla \Pi \in L^1([0, T]; B_{\infty, r}^s) \cap L^\infty([0, T]; L^2)$.

Finally, by interpolation we can pass to the limit in the equations at step n , so we get that $(\rho, u, \nabla \Pi)$ satisfies (1), while continuity in time follows from Proposition 4. \square

5. REMARKS ON THE LIFESPAN OF THE SOLUTION

In this section, we exhibit lower bounds for the lifespan of the solution to the density dependent incompressible Euler equations. We first establish that, like in the homogeneous case, in any dimension, if the initial velocity is of order ε then the lifespan is at least of order ε^{-1} *even if the density is not a small perturbation of a positive real number*. Next we focus on the two-dimensional case: we show in the second part of this section, that for small perturbations of a constant density state, the lifespan tends to be very large. Therefore, for nonhomogeneous incompressible fluids too, the two-dimensional case is somewhat nicer than the general situation.

5.1. The general case. Let ρ_0 , u_0 and f satisfy the assumptions of Theorem 1 or 3. Denote

$$\tilde{u}_0(x) := \varepsilon^{-1} u_0(x) \quad \text{and} \quad \tilde{f}(t, x) := \varepsilon^{-2} f(\varepsilon^{-1} t, x).$$

It is clear that if we set

$$(\rho, u, \nabla \Pi)(t, x) = (\tilde{\rho}, \varepsilon \tilde{u}, \varepsilon^2 \nabla \tilde{\Pi})(\varepsilon t, x)$$

then $(\tilde{\rho}, \tilde{u}, \nabla \tilde{\Pi})$ is a solution to (1) on $[T_*, T^*]$ with data $(\rho_0, \tilde{u}_0, \tilde{f})$ if and only if $(\rho, u, \nabla \Pi)$ is a solution to (1) on $[\varepsilon^{-1} T_*, \varepsilon^{-1} T^*]$ with data (ρ_0, u_0, f) .

Hence, putting together the results of the previous section, we can conclude to the following statement.

Theorem 4. *Let (ρ_0, \tilde{u}_0) satisfy the assumptions of Theorem 1 or 3, and $f \equiv 0$. There exists a positive time T^* depending only on s , N , ρ_* , $\|\rho_0\|_{B_{\infty,1}^0}$ and $\|\tilde{u}_0\|_{B_{\infty,1}^0}$ such that for any $\varepsilon > 0$ the upper bound T_ε^* of the maximal interval of existence for the solution to (1) with initial data $(\rho_0, \varepsilon \tilde{u}_0)$ satisfies*

$$T_\varepsilon^* \geq \varepsilon^{-1} T^*.$$

A similar result holds for the lower bound of the maximal interval of existence.

5.2. The two-dimensional case. Recall that for the homogeneous equations, any solution corresponding to suitably smooth data is global, a fact which relies on the conservation of the vorticity by the flow. Now, in our case, the vorticity equation reads (if $f \equiv 0$)

$$(57) \quad \partial_t \omega + u \cdot \nabla \omega + \nabla b \wedge \nabla \Pi = 0$$

with $b := 1/\rho - 1$ and $\nabla b \wedge \nabla \Pi := \partial_1 b \partial_2 \Pi - \partial_2 b \partial_1 \Pi$.

Owing to the new term involving the pressure and the nonhomogeneity, it is not clear at all that global existence still holds. Nevertheless, we are going to prove that the lifespan may be very large if the nonhomogeneity is small.

To simplify the presentation, we focus on the case where $\rho_0 \in B_{\infty,1}^1(\mathbb{R}^2)$ and $u_0 \in B_{\infty,1}^1(\mathbb{R}^2)$ (note that Corollary 1 ensures that this is not restrictive) and assume, in addition, that $u_0 \in H^1(\mathbb{R}^2)$ (this lower order assumption may be somewhat relaxed too).

We aim at proving the following result.

Theorem 5. *Under the above assumptions there exists a constant c such that if $b_0 := \frac{1}{\rho_0} - 1$ satisfies*

$$(58) \quad \|b_0\|_{B_{\infty,1}^1} \leq c$$

then the lifespan of the solution to the two-dimensional density dependent incompressible Euler equations with initial data (ρ_0, u_0) and no source term is bounded from below by

$$\frac{c}{\|u_0\|_{H^1 \cap B_{\infty,1}^1}} \log \left(1 + \log \frac{c}{\|b_0\|_{B_{\infty,1}^1}} \right).$$

Proof. Let $]T_*, T^*[$ denote the maximal interval of existence of the solution $(\rho, u, \nabla \Pi)$ corresponding to (ρ_0, u_0) . To simplify the presentation, we focus on the evolution for *positive* times.

The key to the proof relies on the fact that in the two-dimensional case, the vorticity equation satisfies (57). Now, it turns out that, as discovered by M. Vishik in [15] and by T. Hmidi and S. Keraani in [12], the norms in Besov spaces *with null regularity index* of solutions to transport equations satisfy better estimates, namely in our case

$$\|\omega(t)\|_{B_{\infty,1}^0} \leq C \left(\|\omega_0\|_{B_{\infty,1}^0} + \int_0^t \|\nabla b \wedge \nabla \Pi\|_{B_{\infty,1}^0} d\tau \right) \left(1 + \int_0^t \|\nabla u\|_{L^\infty} d\tau \right)$$

whereas, according to Proposition 4, the last term has to be replaced with $\exp\left(\int_0^t \|\nabla u\|_{L^\infty} d\tau\right)$ for nonzero regularity exponents.

Therefore, using Inequality (48), we get

$$(59) \quad \|\omega(t)\|_{B_{\infty,1}^0} \leq C \left(\|\omega_0\|_{B_{\infty,1}^0} + \int_0^t \|b\|_{B_{\infty,1}^1} \|\nabla \Pi\|_{B_{\infty,1}^0} d\tau \right) \left(1 + \int_0^t \|\nabla u\|_{L^\infty} d\tau \right)$$

Of course, a basic energy argument leads to

$$(60) \quad \|\omega(t)\|_{L^2} \leq \|\omega_0\|_{L^2} + \int_0^t \|\nabla b\|_{L^\infty} \|\nabla \Pi\|_{L^2} d\tau$$

and it is well-known that for two-dimensional divergence-free vector fields, we have

$$\|\nabla u\|_{L^2} = \|\omega\|_{L^2}.$$

Therefore putting together Inequalities (59) and (60) and bearing in mind Inequality (44) and the energy inequality for u , we get

$$(61) \quad X(t) \leq C \left(X_0 + \int_0^t B \|\nabla \Pi\|_{B_{\infty,1}^0 \cap L^2} d\tau \right) \left(1 + \int_0^t X d\tau \right)$$

with

$$X(t) := \|u(t)\|_{H^1 \cap B_{\infty,1}^1} \quad \text{and} \quad B(t) := \|b(t)\|_{B_{\infty,1}^1}.$$

Bounding B is easy given that

$$\partial_t b + u \cdot \nabla b = 0.$$

Indeed, Inequality (7) ensures that

$$\|b(t)\|_{B_{\infty,1}^1} \leq \|b_0\|_{B_{\infty,1}^1} \exp \left(C \int_0^t \|\nabla u\|_{B_{\infty,1}^0} d\tau \right).$$

Therefore,

$$(62) \quad B(t) \leq B_0 \exp \left(C \int_0^t X d\tau \right).$$

Bounding the pressure term in $B_{\infty,1}^0 \cap L^2$ is our next task. For that, recall that, as

$$\operatorname{div}(a \nabla \Pi) = -\operatorname{div}(u \cdot \nabla u),$$

Lemma 2 guarantees that

$$(63) \quad a_* \|\nabla \Pi\|_{L^2} \leq \|u\|_{L^2} \|\nabla u\|_{L^\infty}.$$

Next, differentiating once the pressure equation and applying again an energy method yields

$$(64) \quad a_* \|\nabla^2 \Pi\|_{L^2} \leq \|\nabla u\|_{L^2} \|\nabla u\|_{L^\infty} + \|\nabla a\|_{L^\infty} \|\nabla \Pi\|_{L^2}.$$

Therefore, combining (63) and (64) and using elementary embedding, we get

$$(65) \quad \|\nabla \Pi\|_{H^1} \leq C \|a\|_{B_{\infty,1}^1} X^2.$$

Note that $\|a\|_{B_{\infty,1}^1}$ and $A := 1 + B$ are of the same order. This will be important in the sequel.

In order to bound the pressure term in $B_{\infty,1}^0$ we shall use the following classical logarithmic interpolation inequality (see e.g. [1], Chap. 2):

$$(66) \quad \|\nabla \Pi\|_{B_{\infty,1}^0} \leq C \|\nabla \Pi\|_{H^1} \log \left(e + \frac{\|\nabla \Pi\|_{B_{\infty,1}^1}}{\|\nabla \Pi\|_{H^1}} \right).$$

In order to estimate $\|\nabla \Pi\|_{B_{\infty,1}^1}$, we use the identity

$$\nabla \Pi = \Delta_{-1} \nabla \Pi + \mathcal{A}(D) \operatorname{div} (u \cdot \nabla u) + \mathcal{A}(D) \operatorname{div} (b \nabla \Pi).$$

with $\mathcal{A}(D) := (-\Delta)^{-1} \nabla (\operatorname{Id} - \Delta_{-1})$.

On the one hand, combining Bony's decomposition with the fact that $\operatorname{div} (u \cdot \nabla u) = \nabla u : \nabla u$, it is easy to show that

$$\|\operatorname{div} (u \cdot \nabla u)\|_{B_{\infty,1}^0} \leq \|u\|_{B_{\infty,1}^1}^2.$$

On the other hand, Proposition 2 guarantees that

$$\|b \nabla \Pi\|_{B_{\infty,1}^1} \leq C \left(\|b\|_{L^\infty} \|\nabla \Pi\|_{B_{\infty,1}^1} + \|\nabla \Pi\|_{L^\infty} \|b\|_{B_{\infty,1}^1} \right).$$

So using the fact that $\mathcal{A}(D)$ (resp. $\mathcal{A}(D) \operatorname{div}$) is a multiplier of degree -1 (resp. 0) away from the origin, we get from Proposition 1,

$$\|\nabla \Pi\|_{B_{\infty,1}^1} \leq C \left(\|\nabla \Pi\|_{L^2} + \|u\|_{B_{\infty,1}^1}^2 + \|\nabla \Pi\|_{L^\infty} \|b\|_{B_{\infty,1}^1} + \|b\|_{L^\infty} \|\nabla \Pi\|_{B_{\infty,1}^1} \right).$$

Note that $\|b(t)\|_{L^\infty}$ is time independent and that $B_{\infty,1}^1 \hookrightarrow L^\infty$. Hence, under assumption (58) with c small enough, the last term may be absorbed by the left-hand side. As regards the last but one term, we use the following interpolation inequality:

$$\|\nabla \Pi\|_{L^\infty} \leq C \|\nabla \Pi\|_{L^2}^{\frac{1}{2}} \|\nabla \Pi\|_{B_{\infty,1}^1}^{\frac{1}{2}}.$$

Combining with Young's inequality, we thus conclude that, under assumption (58), we have

$$\|\nabla \Pi\|_{B_{\infty,1}^1} \leq C \left(\|u\|_{B_{\infty,1}^1}^2 + (1 + \|b\|_{B_{\infty,1}^1}^2) \|\nabla \Pi\|_{L^2} \right).$$

Bounding the last term according to (63), we thus end up with

$$\|\nabla \Pi\|_{B_{\infty,1}^1} \leq CA^2 X^2.$$

Inserting this inequality in (66) and using also (65), one may now conclude that

$$(67) \quad \|\nabla \Pi\|_{B_{\infty,1}^0 \cap L^2} \leq CAX^2 \log(e + B).$$

It is now time to insert Inequalities (62) and (67) in (61); we get

$$(68) \quad X(t) \leq C \left(X_0 + B_0 A_0 \log(e + B_0) \int_0^t e^{C \int_0^\tau X \, d\tau'} X^2 \, d\tau \right) \left(1 + \int_0^t X \, d\tau \right).$$

Let T_0 denote the supremum of times $t \in [0, T^*[$ so that

$$(69) \quad B_0 A_0 \log(e + B_0) \int_0^t e^{C \int_0^\tau X \, d\tau'} X^2 \, d\tau \leq X_0.$$

From (68) and Gronwall's Lemma, we gather that

$$X(t) \leq 2CX_0 e^{2CtX_0} \quad \text{for all } t \in [0, T_0[.$$

Note that this inequality implies that for all $t \in [0, T_0]$, we have

$$\int_0^t e^{C \int_0^\tau X \, d\tau'} X^2 \, d\tau \leq C X_0 \left(e^{4CtX_0} - 1 \right) \exp \left(C \left(e^{2CtX_0} - 1 \right) \right).$$

Therefore, using (69) and a bootstrap argument (based on the continuation theorems that we proved in the previous sections), it is easy to show that T_0 is greater than any time t such that

$$A_0 B_0 \log(e + B_0) \left(e^{4CtX_0} - 1 \right) \exp \left(C \left(e^{2CtX_0} - 1 \right) \right) \leq 1.$$

Taking the logarithm and using that $\log y \leq y - 1$ for $y > 0$, we see that if B_0 is small enough (an assumption which implies in particular that $A_0 \log(e + B_0)$ is of order 1) then the above inequality is satisfied whenever

$$e^{2CtX_0} - 1 \leq \frac{1}{C + 2} \log \left(\frac{1}{2CB_0} \right).$$

This completes the proof of the lower bound for T^* . \square

Remark 5. If ω_0 has more regularity (say $\omega_0 \in C^r$ for some $r \in (0, 1)$) then one may first write an estimate for $\|\omega\|_{L^\infty}$ and next use the classical logarithmic inequality for bounding $\|\nabla u\|_{L^\infty}$ in terms of $\|\omega\|_{L^\infty}$ and $\|\omega\|_{C^r}$. The proof is longer, requires more regularity and, at the same time, the lower bound for the lifespan does not improve.

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